ASYMPTOTICALLY CONICAL CALABI-YAU MANIFOLDS, II

RONAN J. CONLON AND HANS-JOACHIM HEIN

Abstract. Let X be a compact Kähler orbifold of complex dimension ≥ 3, and let D be a suborbifold of X containing the singularities of X that, as a Baily divisor in X, satisfies $-K_X = \alpha[D]$ for some $\alpha \in \mathbb{N}$, $\alpha \geq 2$. Using the results from the first paper of this series, we show that if the pair (X, D) satisfies a mild technical condition and if D admits a Kähler-Einstein orbifold metric of positive scalar curvature, then $X\backslash D$ admits a unique asymptotically conical Calabi-Yau metric in each Kähler class. This refines a theorem of Tian and Yau concerning the existence of Calabi-Yau metrics on non-compact Kähler manifolds.

arXiv:1301.5312v1 [math.DG] 22 Jan 2013

1. Introduction

The purpose of this article is to present a refinement of a theorem by Tian and Yau [40] concerning the existence of complete Ricci-flat Kähler (or Calabi-Yau) metrics on non-compact Kähler manifolds. Before we state our result, let us recall their original theorem. We begin with a definition. (Here and throughout, we write $[D]$ to denote the line orbibundle induced by a Baily divisor D in an orbifold X. The definition of the terms from orbi-geometry here may be found in Appendix A.)

Definition 1.1 ([40, Definition 1.1]). Let X be a compact Kähler orbifold of complex dimension n with singularity set of complex codimension at least two. Suppose that D is a Baily divisor in X. Then:

(a) D is *neat* if no compact holomorphic curve in $X\backslash D$ is homologous to an element in $N_1(D)$, where $N_1(D)$ denotes the abelian group generated by holomorphic curves supported in D.

(b) D is *almost ample* if there exists an integer $m > 0$ such that a basis of $H^0(X, m[D])$ gives a morphism from X into some projective space $\mathbb{P}^N$ that is biholomorphic in a neighbourhood of D.

(c) D is *admissible* if the singular set $\mathrm{Sing}(X)$ of X is contained in D, D is smooth in $X\backslash \mathrm{Sing}(X)$, and if for every $x \in \mathrm{Sing}(X)$, there exists a local uniformising chart $\pi_x : \tilde{U}_x \subset \mathbb{C}^n \longrightarrow U_x$ covering x, such that $\pi_x^{-1}(D)$ is smooth in $\tilde{U}_x$.

With this, Tian and Yau's existence result can be stated as follows.

Theorem 1.2 ([40, Corollary 1.1]). *Let X be a compact Kähler orbifold with singularity set of complex codimension at least two. Let D be a neat, almost ample, and admissible Baily divisor in X, with $-K_X = \alpha[D]$ for some $\alpha > 1$. If D admits a Kähler-Einstein metric with positive scalar curvature, then $X\backslash D$ admits a complete Ricci-flat Kähler metric g with Euclidean volume growth.*

Furthermore, if we denote by $R(g)$ the curvature tensor of g and by $\rho(\,\cdot\,)$ the distance function on $X\backslash D$ from some fixed point with respect to g, then $R(g)$ decays at the order of exactly ρ^{-2} with respect to the g-norm.

The main result of this paper can be summarised as follows. A more precise formulation can be found in Section 1.2.

Theorem 1.3. *Let X be a compact Kähler orbifold of complex dimension $n \geq 3$, and let D be an admissible Baily divisor in X satisfying $-K_X = \alpha[D]$ for some $\alpha \in \mathbb{N}$, $\alpha \geq 2$. Suppose that $K_D\backslash\{0\}$ is smooth and that the pair (X, D) satisfies a mild technical condition relating to how D sits inside X. If D admits a Kähler-Einstein orbifold metric of positive scalar curvature, then for*

Date: January 23, 2013.

all $c > 0$, there exists a unique complete Ricci-flat Kähler metric ω_c in each Kähler class of $X\backslash D$ that resembles (in a precise sense) the scaling $c\omega_0$ of a Ricci-flat Kähler cone metric ω_0 outside a compact set.

By a *Kähler class* on a non-compact complex manifold, we just mean any de Rham cohomology class that can be represented by a (not necessarily complete) positive (1, 1)-form.

Theorem 1.3 can be considered a refinement of Theorem 1.2 in the following ways.

(i) We remove the assumptions of neatness and almost ampleness in Theorem 1.3 that are imposed on the Baily divisor in Theorem 1.2.
(ii) We have a uniqueness statement in Theorem 1.3.
(iii) The statement in Theorem 1.3 concerning the asymptotics of the Ricci-flat Kähler metric is more precise than the corresponding statement in Theorem 1.2.

Note that Tian and Yau themselves remark on [40, p. 30] that the assumption of neatness on D in Theorem 1.2 is probably superfluous.

One might be concerned with the fact that we have "extra" assumptions in the statement of Theorem 1.3, in the sense that we require $K_D\backslash\{0\}$ to be smooth and that we have a "mild technical condition" on the pair (X, D). Neither of these hypotheses are mentioned in the statement of Theorem 1.2. All the same, we think that they are in fact necessary conditions on the pair (X, D), and we hope to show this in a future article.

We note that in [41], van Coevering constructs Calabi-Yau metrics on quasi-projective manifolds $X\backslash D$ under certain technical conditions that are less restrictive than the ones in Theorem 1.2, but still more restrictive than those of Theorem 1.3. However, as it turns out, an extra assumption was made during the proof in [41] that forces $X\backslash D$ to be a crepant resolution of a cone, thus formally reducing [41] to [43]. This shall be discussed in Appendix C.

Let us now give the outline of the remainder of this paper. The rest of this section is devoted to introducing the relevant terminology that we use, recalling our previous results, and in giving precise statements of the results of this paper. In particular, we say what we mean by a Calabi-Yau cone and outline an ansatz (due to Calabi) for their construction. In Section 2, we provide the proof of our main theorem, namely Theorem 1.3. Appendix A then contains all the background material on orbifolds that we assume, and in Appendix B, we prove a version of the Kawamata-Viehweg vanishing theorem for Kähler orbifolds. This result may be of independent interest, and we make use of it in the proof of Theorem 1.3. Finally, as remarked, in Appendix C, we show that the main result of [41] reduces to that of [43].

1.1. **Preliminaries.**

1.1.1. *Riemannian cones.* For us, the definition of a Riemannian cone will take the following form.

Definition 1.4. Let (L, g) be a compact connected Riemannian manifold. The *Riemannian cone* (C, g_0) with link L is defined to be $C := \mathbb{R}^+ \times L$ with metric $g_0 := dr^2 + r^2 g$ up to isometry. The *radius function* r is then characterized intrinsically as the distance from the apex in the metric completion.

Suppose that we are given a Riemannian cone (C, g_0) as above. Let (r, x) be polar coordinates on C, where $x \in L$, and for $t > 0$, define a map

$$\nu_t : L \times [1, 2] \ni (r, x) \mapsto (tr, x) \in L \times [t, 2t].$$

One checks that $\nu_t^*(g_0) = t^2 g_0$ and that $\nu_t^* \circ \nabla_0 = \nabla_0 \circ \nu_t^*$, where ∇_0 denotes the Levi-Civita connection of g_0. Using these facts, one can prove the following basic lemma which will be useful in the proof of Proposition 2.4.

Lemma 1.5. *Suppose that $\alpha \in \Gamma((TC)^{\otimes p} \otimes (T^*C)^{\otimes q})$ satisfies $\nu_t^*(\alpha) = t^k \alpha$ for every $t > 0$ for some $k \in \mathbb{R}$. Then $|\nabla_0^l \alpha|_{g_0} = O(r^{k+p-q-l})$ for all $l \in \mathbb{N}_0$.*

We shall say that "$\alpha = O(r^\lambda)$ with g_0-derivatives" whenever $|\nabla_0^k \alpha|_{g_0} = O(r^{\lambda-k})$ for every $k \in \mathbb{N} \cup \{0\}$. We will then also say that α has "rate at most λ", or sometimes, for simplicity, "rate λ", although it should be understood that the rate is really the infimum of all λ for which this holds.

1.1.2. *Kähler and Calabi-Yau cones.* Boyer-Galicki [10] is a comprehensive reference here.

Definition 1.6. A *Kähler cone* is a Riemannian cone $(C,\, g_0,\, J_0)$ such that g_0 is Kähler, together with a choice of g_0-parallel complex structure J_0. This will in fact often be unique up to sign. We then have a Kähler form $\omega_0(X,Y) = g_0(J_0X, Y)$, and $\omega_0 = \frac{i}{2}\partial\bar{\partial}r^2$ with respect to J_0.

We call a Kähler cone "quasiregular" if the Reeb field $J\partial_r$ on its link generates an S^1-action (and, in particular, "regular" if this S^1-action is free), and "irregular" otherwise.

Definition 1.7. We say that $(C,\, g_0,\, J_0,\, \Omega_0,\, \omega_0)$ is a *Calabi-Yau cone* with *Calabi-Yau cone structure* $(\Omega_0,\, g_0)$ if

(i) $(C,\, g_0,\, J_0)$ is a Ricci-flat Kähler cone with Kähler form ω_0,
(ii) the canonical bundle K_C of C with respect to J_0 is trivial, and
(iii) Ω_0 is a nowhere vanishing section of K_C with $\omega_0^n = i^{n^2}\Omega_0 \wedge \bar{\Omega}_0$. (Here, $n = \dim_{\mathbb{C}} C$).

If a Kähler manifold M has trivial canonical bundle K_M, then we call any nowhere vanishing section of K_M a *holomorphic volume form*. Thus, we may rephrase part (iii) of the above definition as follows: C admits a holomorphic volume form whose norm is constant with respect to g_0.

1.1.3. *Calabi ansatz.* The construction of Calabi-Yau cones, or Sasaki-Einstein manifolds, is in itself a highly nontrivial problem. See Sparks [39] for an excellent recent survey.

The most elementary construction, originating in Calabi's paper [11], states that regular Calabi-Yau cones are classified by Kähler-Einstein Fano manifolds and that quasi-regular Calabi-Yau cones are classified by Kähler-Einstein Fano orbifolds whose local uniformising groups inject into $\mathbb{C}^*$ via the determinant map. This last condition is equivalent to asking for the complement of the zero section of the total space of the canonical orbibundle of the Kähler-Einstein Fano orbifold to be smooth (cf. Lemma A.11). Let D be a Kähler-Einstein Fano orbifold of complex dimension n for which this is the case, and let $K_D^\times$ denote the total space of the blowdown of the zero section of K_D. Then, in a uniformising chart $(\tilde{U},\, \Gamma,\, \varphi)$ of D, the lift to $K_{\tilde{U}}$, the line bundle over $\tilde{U}$ defining K_D, of the Kähler potential of the Calabi-Yau cone metric on $K_D^\times$ resulting from Calabi's construction is $r = \|\cdot\|^{\frac{1}{n+1}}$, where $\|\cdot\|$ denotes the Hermitian norm on $K_{\tilde{U}}$ naturally induced from the Kähler-Einstein metric on $\tilde{U}$ defining the Kähler-Einstein orbifold metric on D. Note that there is a requirement here for the Kähler-Einstein metric on D to be normalised so that the corresponding Einstein constant is $2(n+1)$.

In order to show that $K_D^\times$ is a genuine Calabi-Yau cone with D as above, we must also exhibit a holomorphic volume form on this space whose norm is constant with respect to the aforementioned Calabi-Yau cone metric. Such a holomorphic volume form exists and its pullback to a uniformising chart $(K_{\tilde{U}},\, \Gamma,\, \varphi_{K_{\tilde{U}}})$ of K_D arising from a uniformising chart $(\tilde{U},\, \Gamma,\, \varphi)$ of D is the exterior derivative of the tautological holomorphic $(n,\, 0)$-form σ on $K_{\tilde{U}}$ defined by $\sigma(a)(v_1, \dots, v_n) = a(\pi_*(v_1), \dots, \pi_*(v_n))$. Here, $\pi : K_{\tilde{U}} \longrightarrow \tilde{U}$ is the canonical projection map, a is a point of $K_{\tilde{U}}$, $v_1, \dots, v_n$ are tangent vectors to $K_{\tilde{U}}$ at a, the action of Γ on $K_{\tilde{U}}$ is that induced by the action of Γ on $\tilde{U}$, and $\varphi_{K_{\tilde{U}}}$ is the quotient map associated with this action. In coordinates, this holomorphic volume form has the following description (cf. [29, Proposition 6.1]). A choice of local holomorphic coordinates $(z_1, \dots, z_n)$ on a uniformising chart $(\tilde{U},\, \Gamma,\, \varphi)$ of D induces holomorphic coordinates $(y,\, z_1, \dots, z_n)$ on $K_{\tilde{U}}$. Specifically, $(y,\, z_1, \dots, z_n)$ corresponds to the vector $y\, dz_1 \wedge \dots \wedge dz_n$ in the fibre of $K_{\tilde{U}}$ over $(z_1,\, \dots,\, z_n)$. The lift of the holomorphic volume form on $K_D^\times$ to $K_{\tilde{U}}$ may then be written in terms of these induced coordinates on $K_{\tilde{U}}$ as $dy \wedge dz_1 \wedge \dots \wedge dz_n$. See LeBrun [27, Proposition 3.1] for the computation leading to the fact that the norm of this holomorphic volume form is constant with respect to the Calabi-Yau cone metric.

1.1.4. *Asymptotically conical Calabi-Yau manifolds.* The Calabi-Yau manifolds of interest to us are of the following type.

Definition 1.8. Let (M, Ω, ω) be a complete Ricci-flat Kähler manifold with trivial canonical bundle K_M, holomorphic volume form Ω, and Kähler form ω, and let (C, Ω_0, ω_0) be a Calabi-Yau cone. Then we say that (M, Ω, ω) is an *asymptotically conical (AC) Calabi-Yau manifold with rate* $\lambda < 0$ *modelled on the Calabi-Yau cone* (C, Ω_0, ω_0) if there exists a diffeomorphism $\Phi : C\backslash K \longrightarrow M\backslash K'$ with K, K' compact, such that $\Phi^*\Omega - \Omega_0 = O(r^\lambda)$ and $\Phi^*\omega - \omega_0 = O(r^\lambda)$, both with g_0-derivatives. Here, g_0 is the Kähler cone metric associated to ω_0 and r is the radial coordinate of the cone (C, g_0).

We implicitly only allow for one end in this definition. This is because, by the splitting theorem, an AC Calabi-Yau manifold can only ever have one end anyway.

1.1.5. *Kähler classes on open complex manifolds.* We begin with the definition of a Kähler class on a non-compact complex manifold.

Definition 1.9. Let M be an open complex manifold. Define a *Kähler class* on M to be a cohomology class in $H^2(M, \mathbb{R})$ that can be represented by a positive $(1, 1)$-form. In particular, the associated metric of the representative positive $(1, 1)$-form need not be complete.

Let us next recall from [18] the definition of a μ-almost compactly supported Kähler class.

Definition 1.10. Let M be an open complex manifold, $K \subset M$ a compact set, $C = \mathbb{R}^+ \times L$ a cone with cone metric g_0, and $\Phi : (1, \infty) \times L \longrightarrow M\backslash K$ a diffeomorphism. A Kähler class in $H^2(M, \mathbb{R})$ is called a *μ-almost compactly supported Kähler class* for some given $\mu < 0$, and is said to be *μ-almost compactly supported*, if it can be represented by a (not necessarily complete) Kähler form ω on M such that $\omega - \xi = d\eta$ on $M\backslash K$ with η a smooth real 1-form on $M\backslash K$ and ξ a smooth real $(1, 1)$-form on $M\backslash K$ such that $\Phi^*\xi = O(r^\mu)$ with g_0-derivatives.

1.1.6. *Summary of results from* [18]. The main results from [18] concerning the existence and uniqueness of AC Calabi-Yau metrics can be rephrased in terms of the terminology just introduced as follows.

Theorem 1.11 ([18, Theorems 2.4 & 3.1])**.** *Let M be an open complex manifold of complex dimension $n \geq 3$ such that K_M is trivial. Let Ω be a holomorphic volume form on M and let L be Sasaki-Einstein with associated Calabi-Yau cone (C, Ω_0, ω_0) and radius function r. Suppose that there exists $\lambda < 0$, a compact subset $K \subset M$, and a diffeomorphism $\Phi : (1, \infty) \times L \longrightarrow M\backslash K$ such that*

$$\Phi^*\Omega - \Omega_0 = O(r^\lambda) \quad \textit{with } g_0\textit{-derivatives},$$

where g_0 is the Kähler metric associated to ω_0. Let $\mu < 0$ and assume that $\nu := \max\{\lambda, \mu\} \notin \{-2n, -2, \nu_0 - 2\}$, where $\nu_0 \geq 1$ denotes the smallest growth rate of a pluriharmonic function on C. Then for every $c > 0$, there exists, in each μ-almost compactly supported Kähler class, a unique AC Calabi-Yau metric ω_c satisfying

$$\Phi^*\omega_c - c\omega_0 = O(r^{\max\{-2n, \nu\}}) \quad \textit{with } g_0\textit{-derivatives}.$$

1.2. **Results.** In order to give a precise statement of Theorem 1.3, we introduce the following technical assumption on a pair (X, D), where X is a compact Kähler orbifold of complex dimension n and D is a complex codimension one suborbifold of X satisfying $-K_X = \alpha[D]$ for some $\alpha \in \mathbb{N}$, $\alpha \geq 2$.

Basic Assumption. For every point $x \in D$, there exists a uniformising chart $(\tilde{U}, \Gamma, \varphi)$ of X with $x \in \varphi(\tilde{U})$ that admits holomorphic coordinates $(z_1, \ldots, z_n)$ on $\tilde{U}$ with respect to which

(i) $\varphi^{-1}(D \cap \varphi(\tilde{U})) = \{z_n = 0\}$, and with respect to which

(ii) each element $\gamma \in \Gamma$ takes the form $\begin{pmatrix} A(\gamma) & \star \\ 0 & \det(A(\gamma))^{\frac{1}{\alpha-1}} \end{pmatrix}$ for some $A(\gamma) \in GL(n-1, \mathbb{C})$ depending on γ.

Let us make a few remarks concerning this "basic assumption".

Remark 1.12. First, notice that since D is a suborbifold of X, it is always possible to find a uniformising chart covering each point of D such that condition (i) above holds. However, the elements of Γ do not automatically take the form as laid out in condition (ii) above in the holomorphic coordinates on $\tilde{U}$ fulfilling condition (i).

Remark 1.13. Secondly, since the group Γ in an isotropy chart $(\tilde{U}, \Gamma, \varphi)$ about a point $x \in D$ preserves the tangent plane to D at $\varphi^{-1}(x) \in \tilde{U}$, Γ must necessarily be conjugate to a finite subgroup of $GL(n, \mathbb{C})$ whose elements take the form $\begin{pmatrix} A & \star \\ 0 & e^{i\theta} \end{pmatrix}$ for $A \in GL(n-1, \mathbb{C})$ and $\theta \in \mathbb{R}$. If our basic assumption holds, then condition (ii) above implies that they must in fact be conjugate to a finite subgroup of $GL(n, \mathbb{C})$ whose elements take the more restrictive form $\begin{pmatrix} A & \star \\ 0 & \det(A)^{\frac{1}{\alpha-1}} \end{pmatrix}$ for $A \in GL(n-1, \mathbb{C})$. Thus, we have a necessary condition on the form of the isotropy groups of X at points of D for our basic assumption to hold.

Remark 1.14. Finally, if the pair (X, D) does happen to satisfy the basic assumption, then D acquires an orbifold atlas from X induced from those charts of X on which holomorphic coordinates satisfying conditions (i) and (ii) above exist. This induced atlas is a refinement of that induced on D by those charts of X in which the pre-image of D is smooth (see the comments after Definition A.8).

With the above "basic assumption", we can now state a precise version of Theorem 1.3 as follows.

Theorem 1.15. *Let X be a compact Kähler orbifold of complex dimension $n \geq 3$ and let D be a codimension one complex suborbifold of X with the following properties.*

(a) *D contains the singularities of X.*
(b) *D is Kähler-Einstein Fano.*
(c) *As a Baily divisor in X, D satisfies $-K_X = \alpha[D]$ for some $\alpha \in \mathbb{N}$, $\alpha \geq 2$.*
(d) *The space $K_D \backslash \{0\}$ is smooth.*

Furthermore, assume that the pair (X, D) satisfies the basic assumption. Then in each Kähler class of $X \backslash D$ and for every $c > 0$, there exists a unique AC Calabi-Yau metric ω_c on $X \backslash D$ satisfying

$$\exp^*(\omega_c) - c\omega_0 = O(r^{\max\{-2, -\frac{n}{\alpha-1}\}+\delta}) \quad \textit{with } g_0\textit{-derivatives, for any } \delta > 0.$$

Here, $\exp : N_D \longrightarrow X$ *denotes the restriction to the normal orbibundle N_D of D in X of the exponential map of any background Kähler metric on X, g_0 denotes the pullback to $N_D \backslash \{0\}$ of the Calabi ansatz Ricci-flat Kähler cone metric on $K_D^\times$ via the covering map defined in Corollary 2.3, and ω_0 and r denote the Kähler form and radial cone coordinate associated to g_0 respectively.*

Note that, since $X \backslash D$ is smooth by condition (a), condition (c) is equivalent to saying that $X \backslash D$ admits a holomorphic volume form blowing up to order α along D when the holomorphic volume form is viewed as a section of K_X. Also note that, by Lemma A.11, condition (d) is equivalent to requiring that each local uniformising group Γ of D injects into $\mathbb{C}^*$ via the map $\det : \Gamma \longrightarrow \mathbb{C}^*$. We further remark that the existence of a Calabi-Yau cone metric on $K_D^\times$ is guaranteed by the Calabi ansatz by virtue of conditions (b) and (d) (cf. Section 1.1.3). Finally, we note that, by the tubular neighbourhood theorem for suborbifolds (cf. Example A.22), the map $\exp : N_D \longrightarrow X$ is indeed a diffeomorphism onto its image in some neighbourhood of the zero section of N_D.

With regards to the results of our first paper [18], notice that the rates of convergence of the AC Calabi-Yau metrics constructed on smoothings of complete intersection Calabi-Yau cones in [18,

§5] are sharper than those derived from an application of Theorem 1.15 to the smoothings. The difference in rates here is a consequence of the choice of diffeomorphism in each case. In [18, §5], we use the normal projection map, whereas in Theorem 1.15, we use the exponential map of any background Kähler metric. This latter diffeomorphism is clearly more generic than the former, and so it is not a surprise that the rates of convergence for the latter are not as sharp as those for the former. Theorem 1.15 is more general than the construction of AC Calabi-Yau manifolds outlined in [18, §5], but the price we have to pay for this greater degree of generality is a loss in precision of the picture of how the AC Calabi-Yau metrics look like at infinity.

1.3. **Acknowledgements.** This paper comprises the second of two papers studying AC Calabi-Yau manifolds, the first of which is [18]. Both of these papers developed out of RJC's doctoral dissertation [17] at Imperial College London. He would like to thank his supervisor, Mark Haskins, for his guidance and constant support and encouragement, and Gilles Carron and Dominic Joyce for useful remarks on a preliminary version of his thesis. HJH acknowledges postdoctoral support under EPSRC Leadership Fellowship EP/G007241/1. We wish to thank MPIM and HIM Bonn for excellent working conditions during our stay in Bonn in Fall 2011 and the members of the Imperial geometry group for many helpful conversations. Finally, we must acknowledge an intellectual debt of gratitude to Craig van Coevering, whose series of articles [41, 42, 43, 44] inspired much of our research.

2. Optimal existence on quasi-projective manifolds

2.1. **Outline of proof.** Let X be a compact Kähler orbifold and let D be a codimension one complex suborbifold of X containing the singularities of X that, as a Baily divisor in X, satisfies $-K_X = \alpha[D]$ for some $\alpha \in \mathbb{N}$, $\alpha \geq 2$. In this section, we provide the proof of our main theorem, Theorem 1.15. The outline of our proof is as follows.

We first prove an orbifold version of the adjunction formula for X and D, presented in terms of the Poincaré residue. In order to make this work, we must make a technical assumption on the pair (X, D). This is precisely the basic assumption of Section 1.2. We follow this up with a simple lemma that states that the complement of the zero section of the holomorphic normal orbibundle of a codimension one complex suborbifold is isomorphic to the complement of the zero section of its dual, and notes that tensor products of the normal orbibundle with itself can be used to define a covering map. Using these results, we then construct a covering map from $N_D\backslash\{0\}$ to $K_D\backslash\{0\}$ under the hypothesis that the pair (X, D) satisfies the basic assumption. (Here and throughout, we use N_D and N_D^* to denote the normal and conormal orbibundles of D in X respectively). This is presented in Corollary 2.3. Assuming moreover that D is Kähler-Einstein Fano and that $K_D\backslash\{0\}$ is smooth, we can use this covering map to lift the Calabi-Yau cone structure (given to us by the Calabi ansatz) on $K_D^{\times}$ to a Calabi-Yau cone structure (Ω_0, g_0) on $N_D\backslash\{0\}$. We can then compute the rate of convergence of the holomorphic volume forms on $X\backslash D$ and on $N_D\backslash\{0\}$ with respect to g_0. This is precisely what we do in the next step of our proof, namely Proposition 2.4. With this estimate, we are able to immediately apply the existence result from our previous paper [18] to assert, for each $c > 0$, the existence of a unique AC Calabi-Yau metric in each μ-almost compactly supported Kähler class of $X\backslash D$ with $\mu < 0$, asymptotic to the Calabi-Yau cone metric cg_0. However, in order to complete the proof of Theorem 1.15, we must show that *every* Kähler class on $X\backslash D$ is μ-almost compactly supported for some $\mu < 0$. What this essentially comes down to is proving that every Kähler class on $X\backslash D$ can be represented by the restriction of a closed real $(1, 1)$-form on X. We show this by first appealing to the Kawamata-Viehweg vanishing theorem for Kähler orbifolds proved in Appendix B to assert that $h^{0,2}(X) = 0$, before applying the Gysin sequence for orbifolds (cf. Proposition A.26). This will in fact show the stronger result that under the hypotheses of Theorem 1.15, every cohomology class in $H^2(X\backslash D, \mathbb{R})$ can be represented by the restriction of a closed real $(1, 1)$-form on X. We outline the details of this last step in Sections 2.4-2.6.

Note that, as we shall always be working with a complex suborbifold lying inside a Kähler orbifold, we will always treat both the normal and conormal orbibundles of the suborbifold as holomorphic orbibundles here.

2.2. **Construction of a covering map.** We now proceed with our first proposition, an analogue of the adjunction formula for orbifolds.

Proposition 2.1. *Let $(X, \mathcal{U})$ be a compact Kähler orbifold of complex dimension n with orbifold atlas $\mathcal{U}$, and let D be a codimension one complex suborbifold of X containing the singularities of X that, as a Baily divisor in X, satisfies $-K_X = \alpha[D]$ for some $\alpha \in \mathbb{N}$, $\alpha \geq 2$. Furthermore, assume that the basic assumption holds for the pair (X, D). Then there exists a (non-canonical) isomorphism $K_D \cong (\alpha - 1)N_D^*$ determined by a choice of trivialising section of $K_{X\backslash D}$ that, as a section of K_X, blows up to order α along D.*

The precise definition of the isomorphism will be given in the proof below.

Proof of Proposition 2.1. First, let $\mathcal{V} := \{(\tilde{U}_i, \Gamma_i, \varphi_i)\}_{i\in I} \subset \mathcal{U}$ be an indexing of all the uniformising charts $(\tilde{U}, \Gamma, \varphi)$ of X for which $\varphi^{-1}(D \cap \varphi(\tilde{U}))$ is non-empty and for which holomorphic coordinates exist on $\tilde{U}$ satisfying conditions (i) and (ii) of the basic assumption. Set $\tilde{D}_i := \varphi_i^{-1}(D \cap \varphi_i(\tilde{U}_i))$. Then, for each $i \in I$, we see that there exists a holomorphic function $\tilde{s}_i$ on $\tilde{U}_i$ with $d\tilde{s}_i|_{\tilde{D}_i} \neq 0$ such that $\tilde{D}_i = \{x \in \tilde{U}_i : \tilde{s}_i(x) = 0\}$, and that the collection $\mathcal{W} := \{(\tilde{D}_i, \Gamma_i, \varphi_i|_{\tilde{D}_i})\}_{i\in I}$ comprises

an orbifold atlas for D. As remarked in Appendix A, for each $i \in I$, the vector bundles on the uniformising chart $(\tilde{D}_i, \Gamma_i, \varphi_i|_{\tilde{D}_i})$ of D that define the normal orbibundle N_D and the conormal orbibundle N_D^* of D in X are given by the normal bundle $N_{\tilde{D}_i}$ and the conormal bundle $N^*_{\tilde{D}_i}$ of $\tilde{D}_i$ in $\tilde{U}_i$ respectively. For each $i \in I$, we let $\frac{\partial}{\partial \tilde{s}_i}$ denote the local spanning section of $N_{\tilde{D}_i}$ on which $d\tilde{s}_i|_{\tilde{D}_i}$ evaluates to one. The section $\frac{\partial}{\partial \tilde{s}_i}$, together with $d\tilde{s}_i|_{\tilde{D}}$, then give us local trivialisations of $N_{\tilde{D}_i}$ and $N^*_{\tilde{D}_i}$ respectively over $(\tilde{D}_i, \Gamma_i, \varphi_i|_{\tilde{D}_i})$, so that the orbifold atlas $\{(\tilde{D}_i, \Gamma_i, \varphi_i|_{\tilde{D}_i})\}_{i\in I}$ of D is in fact fine for both N_D and N_D^*.

Next, let Ω be any trivialising section of $K_{X\setminus D}$, that, as a section of K_X, blows up to order α along D. By definition, Ω has the following two properties.

(a) If $(\tilde{U}, \Gamma, \varphi)$ is any uniformising chart of X and $\tilde{D} := \varphi^{-1}(\varphi(\tilde{U})\cap D)$, then $\varphi^*\Omega$ is a Γ-invariant holomorphic volume form on $\tilde{U}\setminus\tilde{D}$ that blows up to order α along $\tilde{D}$.
(b) If $\lambda : (\tilde{U}, \Gamma, \varphi) \longrightarrow (\tilde{U}', \Gamma', \varphi')$ is an injection between any two uniformising charts of X, and $\tilde{D}$ is as before, then $\lambda^*(((\varphi')^*\Omega)|_{\lambda(\tilde{U}\setminus\tilde{D})}) = (\varphi^*\Omega)|_{\tilde{U}\setminus\tilde{D}}$.

In particular, from property (a) above, we see that in each uniformising chart $(\tilde{U}_i, \Gamma_i, \varphi_i)$ in the collection $\mathcal{V}$, the form $\varphi_i^*\Omega$ defines a Γ_i-invariant holomorphic volume form blowing up to order α along $\tilde{D}_i$. The form $\varphi_i^*\Omega$ also gives rise to a holomorphic volume form on (the whole of) $\tilde{U}_i$, namely $\tilde{s}_i^\alpha\varphi_i^*\Omega$.

With these preliminaries out of the way, consider now, for each $i \in I$, the following fibrewise linear isomorphism ψ_i, defined at a point $z \in \tilde{D}_i$ in the uniformising chart $(\tilde{D}_i, \Gamma_i, \varphi_i|_{\tilde{D}_i})$ of D by

$$\begin{aligned} &\psi_i : (\alpha-1)N^*_{\tilde{D}_i} \longrightarrow K_{\tilde{D}_i} \\ &\psi_i((z, \lambda\, d\tilde{s}_i^{\otimes(\alpha-1)}|_z)) = (z, \lambda\, ((\tilde{s}_i^\alpha\varphi_i^*\Omega) \lrcorner \frac{\partial}{\partial\tilde{s}_i}|_z)|_{\tilde{D}_i}) \quad \text{for } \lambda \in \mathbb{C}. \end{aligned} \tag{2.1}$$

For each $i \in I$, one can show that this map is independent of the choice of defining function for $\tilde{D}_i$ in $\tilde{U}_i$, hence is well-defined. For each $i \in I$ therefore, we have a well-defined isomorphism $\psi_i : (\alpha-1)N^*_{\tilde{D}_i} \simeq K_{\tilde{D}_i}$. We wish to show that the collection $\{\psi_i\}_{i\in I}$ induces an isomorphism between the orbibundles $(\alpha-1)N_D^*$ and K_D, the existence of which is the content of the proposition. In order to prove this, it suffices to show that, for each $i \in I$, the map ψ_i is equivariant with respect to the induced action of Γ_i on $(\alpha-1)N_{\tilde{D}_i}$ and $K_{\tilde{D}_i}$, and that the collection $\{\psi_i\}_{i\in I}$ patches together on the overlaps of the uniformising charts of D. Let us first prove the former assertion, before going through the details of the proof of the latter.

Fix $i \in I$, let $(\tilde{U}_i, \Gamma_i, \varphi_i)$, $\tilde{D}_i$, and $\tilde{s}_i$ be as above, and let $f_\gamma : \tilde{U}_i \longrightarrow \tilde{U}_i$ denote the biholomorphism induced by some $\gamma \in \Gamma_i$. By construction of the collection $\mathcal{V}$, we know that there exists holomorphic coordinates $(z_1, \ldots, z_n)$ on $\tilde{U}_i$ for which $\tilde{D}_i = \{z_n = 0\}$ and in which f_γ takes the form $\begin{pmatrix} A & \star \\ 0 & \det(A)^{\frac{1}{\alpha-1}} \end{pmatrix}$ for some $A(\gamma) \in GL(n-1, \mathbb{C})$. Henceforth working in these coordinates, we can write $\tilde{s}_i = h.z_n$ for some nowhere vanishing holomorphic function h on $\tilde{U}_i$, so that $d\tilde{s}_i = h.\, dz_n$ along $\tilde{D}_i$. The left-hand side of (2.1) thus transforms as

$$\begin{aligned} (z, d\tilde{s}_i^{\otimes(\alpha-1)}) = (z, h(z)^{(\alpha-1)}.\, dz_n^{\otimes(\alpha-1)}) &\longmapsto (f_{\gamma^{-1}}(z), \det(A)h(z)^{(\alpha-1)}dz_n^{\otimes(\alpha-1)}) \\ &= (f_{\gamma^{-1}}(z), \det(A)\left(\frac{h(z)}{h(f_{\gamma^{-1}}(z))}\right)^{(\alpha-1)} d\tilde{s}_i^{\otimes(\alpha-1)}) \end{aligned} \tag{2.2}$$

at a point $z \in \tilde{D}_i$, under the induced action of γ. As for the right-hand side of (2.1), we have $h^{-1}\cdot\frac{\partial}{\partial z_n}$ lying in the same equivalence class of $N_{\tilde{D}_i}$ as $\frac{\partial}{\partial\tilde{s}_i}$, as well as Γ_i-invariance of $\varphi_i^*\Omega$ on $\tilde{U}_i\setminus\tilde{D}_i$. So the

form $((\tilde{s}_i^\alpha \varphi^*\Omega) \lrcorner \frac{\partial}{\partial \tilde{s}_i})|_{\tilde{D}_i}$ must transform as

$$\begin{aligned}
(z,\, ((\tilde{s}_i^\alpha \varphi^*\Omega) \lrcorner \frac{\partial}{\partial \tilde{s}_i})|_{\tilde{D}_i}) &= (z,\, h(z)^{(\alpha-1)}((z_n^\alpha \varphi^*\Omega) \lrcorner \frac{\partial}{\partial z_n})|_{\tilde{D}_i}) \\
&\longmapsto (f_{\gamma^{-1}}(z),\, h(z)^{(\alpha-1)}((\det(A)^{\frac{1}{\alpha-1}}(z_n^\alpha \varphi^*\Omega)) \lrcorner \left(\det(A)^{-\frac{1}{\alpha-1}}\frac{\partial}{\partial z_n}\right))|_{\tilde{D}_i}) \\
&= (f_{\gamma^{-1}}(z),\, \det(A)h(z)^{(\alpha-1)}((z_n^\alpha \varphi^*\Omega) \lrcorner \frac{\partial}{\partial z_n})|_{\tilde{D}_i}) \\
&= (f_{\gamma^{-1}}(z),\, \det(A)\left(\frac{h(z)}{h(f_{\gamma^{-1}}(z))}\right)^{(\alpha-1)}((\tilde{s}_i^\alpha \varphi^*\Omega) \lrcorner \frac{\partial}{\partial \tilde{s}_i})|_{\tilde{D}_i})
\end{aligned} \tag{2.3}$$

at a point $z \in \tilde{D}_i$, under the action of γ. Here we have used the fact that $(z_1, \ldots, z_{n-1})$ define holomorphic coordinates on $\tilde{D}_i$ and the obvious fact that at any $z \in \tilde{D}_i$,

$$((z_n^\alpha \varphi_i^*\Omega) \lrcorner \frac{\partial}{\partial z_n})|_{\tilde{D}_i} = ((z_n^\alpha \varphi_i^*\Omega) \lrcorner \left(\frac{\partial}{\partial z_n} + V\right))|_{\tilde{D}_i} \quad \text{for any } V \in T_z\tilde{D}_i.$$

Comparing (2.2) and (2.3), one sees that, for each i, the map ψ_i is indeed Γ_i-equivariant, as claimed.

As for showing that our isomorphism patches together on overlaps, suppose that for some $i,\, j \in I$, $\lambda_{ji} : (\tilde{U}_i,\, \Gamma_i,\, \varphi_i) \longrightarrow (\tilde{U}_j,\, \Gamma_j,\, \varphi_j)$ is an injection. Choose holomorphic coordinates $(z_1, \ldots, z_n)$ on $\tilde{U}_i$ and $(w_1, \ldots, w_n)$ on $\tilde{U}_j$ with respect to which $\tilde{D}_i = \{z_n = 0\}$ and $\tilde{D}_j = \{w_n = 0\}$, and with respect to which $\varphi_i^*\Omega = \frac{dz_1 \wedge \ldots \wedge dz_n}{z_n^\alpha}$ and $\varphi_i^*\Omega = \frac{dw_1 \wedge \ldots \wedge dw_n}{w_n^\alpha}$. Writing λ_{ji} in these coordinates, we have

$$\lambda_{ji}((z_1, \ldots, z_n)) = (\lambda_{ji}^1, \ldots, \lambda_{ji}^n),$$

where $\lambda_{ji}^k = \lambda_{ji}^k(z_1, \ldots, z_n)$ for $k = 1, \ldots, n$, and where $\lambda_{ji}^n(z_1, \ldots, z_{n-1}, 0) = 0$. It follows from this last equality that the exterior derivative of λ_{ji} restricted to $\tilde{D}_i$, that is, $d\lambda_{ji}|_{\tilde{D}_i}$, takes the form

$$d\lambda_{ji}|_{\tilde{D}_i = \{z_n = 0\}} = \begin{pmatrix} A & \star \\ 0 & \frac{\partial \lambda_{ji}^n}{\partial z_n} \end{pmatrix}. \tag{2.4}$$

Now, from property (b) above, we know that $\lambda_{ji}^*\left(\frac{dw_1 \wedge \ldots \wedge dw_n}{w_n^\alpha}\right) = \frac{dz_1 \wedge \ldots \wedge dz_n}{z_n^\alpha}$ on $\tilde{U}_i \backslash \tilde{D}_i$. This in particular implies that

$$\det(d\lambda_{ji}) = \left(\frac{\lambda_{ji}^n}{z_n}\right)^\alpha \quad \text{on } \tilde{U}_i \backslash \tilde{D}_i.$$

Now, looking at the form of (2.4), it is clear that $\det(d\lambda_{ji}) \longrightarrow \det(A) \cdot \frac{\partial \lambda_{ji}^n}{\partial z_n}$ as $z_n \longrightarrow 0$. Thus, we find that along $\tilde{D}_i$,

$$\det(A) = \left(\frac{\partial \lambda_{ji}^n}{\partial z_n}\right)^{\alpha-1}.$$

From this, the commutativeness of the following diagram at any point $z \in \tilde{D}_i$, and hence the agreement of the isomorphisms ψ_i and ψ_j on the overlap of the charts $(\tilde{D}_i,\, \Gamma_i,\, \varphi_i|_{\tilde{D}_i})$ and $(\tilde{D}_j,\, \Gamma_j,\, \varphi_j|_{\tilde{D}_j})$ of D, is now immediate.

$$\begin{array}{ccc}
(dw_n)^{(\alpha-1)} \in (N_{\tilde{D}_j}^*)^{(\alpha-1)}|_{\lambda_{ji}(z)} & \xrightarrow{\lambda_{ji}^*} & \left(\frac{\partial \lambda_{ji}^n}{\partial z_n}\right)^{\alpha-1}(dz_n)^{(\alpha-1)} \in (N_{\tilde{D}_i}^*)^{(\alpha-1)}|_z \\
\Big\downarrow {\scriptstyle \psi_j} & & \Big\downarrow {\scriptstyle \psi_i} \\
dw_1 \wedge \ldots \wedge dw_{n-1} \in K_{\tilde{D}_j}|_{\lambda_{ji}(z)} & \xrightarrow[\lambda_{ji}^*]{} & \det(A)\, dz_1 \wedge \ldots dz_{n-1} \in K_{\tilde{D}_i}|_z
\end{array}$$

This completes the proof of the proposition. □

Next, we have:

Lemma 2.2. *Let $(X, \mathcal{U})$ be a compact Kähler orbifold and let D be a codimension one complex suborbifold of X. Then there exists an isomorphism $N_D\backslash\{0\} \cong N^*_D\backslash\{0\}$. Moreover, for any $\beta \in \mathbb{N}$, there exists an unbranched β-fold covering map $N^*_D\backslash\{0\} \longrightarrow (\beta N^*_D)\backslash\{0\}$.*

The definitions of both maps in question here are also given in the proof.

Proof of Lemma 2.2. Let $(\tilde{U}, \Gamma, \varphi)$ be any uniformising chart of X in which $\tilde{D} := \varphi^{-1}(D \cap \varphi(\tilde{U}))$ is non-empty, smooth, and equal to $\{x \in \tilde{U} : \tilde{s}(x) = 0\}$ for some holomorphic function $\tilde{s}$ on $\tilde{U}$ with $d\tilde{s}|_{\tilde{D}} \neq 0$. Then, at any point $z \in \tilde{D}$, the isomorphism between $N_D\backslash\{0\}$ and $N^*_D\backslash\{0\}$ can be written as the fibre-wise map

$$\begin{aligned} &N_{\tilde{D}}\backslash\{0\}|_z \longrightarrow N^*_{\tilde{D}}\backslash\{0\}|_z \\ &(z,\, \lambda.\,\frac{\partial}{\partial\tilde{s}}) \longmapsto (z,\, \frac{1}{\lambda}.\,d\tilde{s}) \quad \text{for } \lambda \in \mathbb{C}^*, \end{aligned} \tag{2.5}$$

whereas the β-fold covering map $N^*_D\backslash\{0\} \longrightarrow (\beta N^*_D)\backslash\{0\}$ can be written as

$$\begin{aligned} &N^*_{\tilde{D}}\backslash\{0\}|_z \longrightarrow (\beta N^*_{\tilde{D}})\backslash\{0\}|_z \\ &(z,\, \lambda.\,d\tilde{s}) \longmapsto (z,\, (\lambda.\,d\tilde{s})^{\otimes\beta}) \quad \text{for } \lambda \in \mathbb{C}^*. \end{aligned} \tag{2.6}$$

Here, $N_{\tilde{D}}$ is the holomorphic normal bundle and $N^*_{\tilde{D}}$ is the holomorphic conormal bundle of $\tilde{D}$ in $\tilde{U}$, and $\frac{\partial}{\partial\tilde{s}}$ is the local section of $N_{\tilde{D}}$ on which $d\tilde{s}|_{\tilde{D}}$ evaluates to one. One can check that the maps (2.5) and (2.6) are both Γ-equivariant and patch together on the overlaps of the uniformising charts of D that are induced from those uniformising charts of X with the same properties as $(\tilde{U}, \Gamma, \varphi)$ above. Hence we do indeed have an isomorphism $N_D\backslash\{0\} \simeq N^*_D\backslash\{0\}$ and a covering map $N^*_D\backslash\{0\} \longrightarrow (\beta N^*_D)\backslash\{0\}$, as asserted. $\square$

As a consequence of Proposition 2.1 and Lemma 2.2, we obtain the following.

Corollary 2.3. *Let $(X, \mathcal{U})$ be a compact Kähler orbifold and let D be a codimension one complex suborbifold of X containing the singularities of X that, as a Baily divisor in X, satisfies $-K_X = \alpha[D]$ for some $\alpha \in \mathbb{N}$, $\alpha \geq 2$. Furthermore, assume that the pair (X, D) satisfies the basic assumption.*

Then there exists an unbranched holomorphic $(\alpha-1)$-fold covering map $\phi_\Omega : N_D\backslash\{0\} \longrightarrow K_D\backslash\{0\}$, which depends only upon a choice of holomorphic volume form Ω on $X\backslash D$ blowing up to order α along D (when considered a section of K_X), that can be realised at a point $z \in \tilde{D} := \varphi^{-1}(D \cap \varphi(\tilde{U}))$ in a uniformising chart $(\tilde{D}, \Gamma, \varphi|_{\tilde{D}})$ of D induced by a uniformising chart $(\tilde{U}, \Gamma, \varphi)$ of X for which $\tilde{D} \neq \emptyset$ and for which holomorphic coordinates exist on $\tilde{U}$ satisfying conditions (i) and (ii) of the basic assumption, as the Γ-equivariant composition

$$\begin{aligned} &(N_{\tilde{D}}|_z)\backslash\{0\}) \longrightarrow (N^*_{\tilde{D}}|_z)\backslash\{0\} \longrightarrow ((\alpha-1)N^*_{\tilde{D}}|_z)\backslash\{0\} \longrightarrow (K_{\tilde{D}}|_z)\backslash\{0\} \\ &(z,\, \lambda.\,\frac{\partial}{\partial\tilde{s}}) \longmapsto (z,\, \lambda^{-1}.\,d\tilde{s}) \longmapsto (z,\, \lambda^{1-\alpha}.\,d\tilde{s}^{\otimes(\alpha-1)}) \longmapsto (z,\, \lambda^{1-\alpha}.\,((\tilde{s}^\alpha\varphi^*\Omega)\lrcorner\frac{\partial}{\partial\tilde{s}})|_{\tilde{D}}) \quad \textit{for } \lambda \in \mathbb{C}^*. \end{aligned} \tag{2.7}$$

Here, $\tilde{s}$ is a holomorphic function on $\tilde{U}$ vanishing to order one along $\tilde{D}$ and $\frac{\partial}{\partial\tilde{s}}$ is the section of $N_{\tilde{D}}$ on which $d\tilde{s}|_{\tilde{D}}$ evaluates to one.

2.3. **Asymptotics of the holomorphic volume forms.** If the pair (X, D) does happen to satisfy the hypotheses of Corollary 2.3, and $K_D^\times$ is moreover a Calabi-Yau cone, then we can use the covering map (2.7) to lift the Calabi-Yau cone structure on $K_D^\times$ to one on $N_D\backslash\{0\}$. This is precisely what we do in the following all-important estimate that will eventually yield Theorem 1.15.

Proposition 2.4. *Let $(X, \mathcal{U})$ be a compact Kähler orbifold of complex dimension n and let D be a codimension one complex suborbifold of X with the following properties.*

(a) *D contains the singularities of X.*
(b) *D is Kähler-Einstein Fano.*
(c) *As a Baily divisor in X, D satisfies $-K_X = \alpha[D]$ for some $\alpha \in \mathbb{N}$, $\alpha \geq 2$.*
(d) *The space $K_D \backslash \{0\}$ is smooth.*

Assume that the pair (X, D) satisfies the basic assumption and let $\exp : N_D \longrightarrow X$ *denote the restriction to N_D of the exponential map of any background Kähler metric g on X, where we identify N_D with $(T^{1,0}D)^{\perp}$, the g-orthogonal complement of $T^{1,0}D$ in $T^{1,0}X|_D$. Then, in the neighbourhood of the zero section of N_D on which* $\exp$ *is a diffeomorphism, we have*

$$\exp^*(\Omega) - \Omega_0 = O(r^{-\frac{n}{\alpha-1}}) \quad \textit{with } g_0\textit{-derivatives,}$$

where Ω denotes any holomorphic volume form on $X \backslash D$ blowing up to order α along D (when considered a section of K_X), g_0 denotes the pullback to $N_D \backslash \{0\}$ of the Calabi ansatz Ricci-flat Kähler cone metric on $K_D^{\times}$ via the holomorphic covering map $\phi_\Omega : N_D \backslash \{0\} \longrightarrow K_D \backslash \{0\}$ of Corollary 2.3 determined by Ω, Ω_0 denotes $(1-\alpha)^{-1}$ times the ϕ_Ω-pullback of the tautological holomorphic volume form on $K_D^{\times}$, and r denotes the radial cone coordinate of g_0.

Note that conditions (a) and (c) here imply that the holomorphic volume form Ω on $X \backslash D$ exists. See also the comments after Theorem 1.15 for some further remarks.

Proof of Proposition 2.4. For clarity, we break the proof down into several parts.

The set-up and notation. First, let z be any point of D and let $(\tilde{U}, \Gamma, \varphi)$ be a uniformising chart of X with $z \in \varphi(\tilde{U})$ satisfying the conditions of the basic assumption. Pick holomorphic coordinates $(z_1, \ldots, z_n)$ on $\tilde{U}$ with respect to which $\tilde{D} := \varphi^{-1}(D \cap \varphi(\tilde{U})) = \{z_n = 0\}$ and with respect to which the Γ-invariant holomorphic volume form $\tilde{\Omega} := \varphi^*\Omega|_{\tilde{U} \backslash \tilde{D}}$ on $\tilde{U} \backslash \tilde{D}$ takes the form

$$\tilde{\Omega} := (\varphi^*\Omega)|_{\tilde{U} \backslash \tilde{D}} = \frac{dz_1 \wedge \ldots \wedge dz_n}{z_n^{\alpha}}. \tag{2.8}$$

(Note that this particular choice of holomorphic coordinates on $\tilde{U}$ may not necessarily satisfy condition (ii) of the basic assumption).

Next, consider the normal orbibundle N_D of D in X. Using the Kähler metric g, it may be identified with $(T^{1,0}D)^{\perp}$, the g-orthogonal complement of $T^{1,0}D$ in $T^{1,0}X|_D$. Now, the total space of the orbibundles $(T^{1,0}D)^{\perp}$ and N_D are themselves orbifolds, and over the open set $\varphi(\tilde{D})$ of D, they admit uniformising charts $((T^{1,0}\tilde{D})^{\perp}, \Gamma^{\perp}, \varphi^{\perp})$ and $(N_{\tilde{D}}, \Gamma_{N_{\tilde{D}}}, \varphi_{N_{\tilde{D}}})$ respectively, the constituents of which are as follows. The open set $(T^{1,0}\tilde{D})^{\perp}$ is the orthogonal complement of $T^{1,0}\tilde{D}$ in $T^{1,0}\tilde{U}|_{\tilde{D}}$ with respect to the Kähler metric on $\tilde{U}$ defining g, the open set $N_{\tilde{D}}$ is the normal bundle of $\tilde{D}$ in $\tilde{U}$, the action of the local uniformising groups $\Gamma^{\perp}$ and $\Gamma_{N_{\tilde{D}}}$ is that induced by Γ on $(T^{1,0}\tilde{D})^{\perp}$ and $N_{\tilde{D}}$ respectively, and the maps $\varphi^{\perp}$ and $\varphi_{N_{\tilde{D}}}$ are the projections onto the quotients $(T^{1,0}\tilde{D})^{\perp}/\Gamma^{\perp}$ and $N_{\tilde{D}}/\Gamma_{N_{\tilde{D}}}$ respectively. By abuse of notation, we denote by $\frac{\partial}{\partial z_n}|_{\tilde{D}}$ the section of $(T^{1,0}\tilde{D})^{\perp}$, and by $\left[\frac{\partial}{\partial z_n}|_{\tilde{D}}\right]$ the section of $N_{\tilde{D}}$, on which $dz_n|_{\tilde{D}}$ evaluates to one. Since we have trivialising sections of $(T^{1,0}\tilde{D})^{\perp}$ and $N_{\tilde{D}}$ here, we have induced holomorphic splittings $N_{\tilde{D}}, (T^{1,0}\tilde{D})^{\perp} \simeq \tilde{D} \times \mathbb{C}$. We introduce holomorphic coordinates $(w_1, \ldots, w_n)$ on $N_{\tilde{D}}$ corresponding to its particular splitting, where w_k denotes the z_k coordinate on the base $\tilde{D}$ if $k \neq n$, and w_n denotes the coordinate in the fibre induced by $\left[\frac{\partial}{\partial z_n}|_{\tilde{D}}\right]$. In these coordinates therefore, the n-tuple $(w_1, \ldots, w_n)$ represents the vector $w_n \left[\frac{\partial}{\partial z_n}|_{\tilde{D}}\right]$ in the fibre of $N_{\tilde{D}}$ over the point $(w_1, \ldots, w_{n-1}, 0) = (z_1, \ldots, z_{n-1}, 0) \in \tilde{D}$. Now, recall that we have an identification $N_D \simeq (T^{1,0}D)^{\perp}$ induced by g. Explicitly, over $\tilde{D}$, we identify the line bundle $N_{\tilde{D}}$ with $(T^{1,0}\tilde{D})^{\perp}$ via $\left[\frac{\partial}{\partial z_n}|_{\tilde{D}}\right] \longmapsto \frac{\partial}{\partial z_n}|_{\tilde{D}}$. Under this identification, the coordinates

we have just introduced on $N_{\tilde{D}}$ also serve to provide coordinates on $(T^{1,0}\tilde{D})^{\perp}$. More precisely, the n-tuple $(w_1, \ldots, w_n)$ can also be taken to represent the vector $w_n\frac{\partial}{\partial z_n}$ in the fibre of $(T^{1,0}\tilde{D})^{\perp}$ over $(w_1, \ldots, w_{n-1}, 0) = (z_1, \ldots, z_{n-1}, 0) \in \tilde{D}$.

Turning our attention now to K_D, the total space of this orbibundle also admits a uniformising chart over the open subset $\varphi(\tilde{D})$ of D. Indeed, such a chart is given by $(K_{\tilde{D}}, \Gamma|_{\tilde{D}}, \varphi_{K_{\tilde{D}}})$, where the action of $\Gamma|_{\tilde{D}}$ on $K_{\tilde{D}}$ is just the induced one, and $\varphi_{K_{\tilde{D}}}$ is the associated quotient projection map. What is important for us to take note of here is that the line bundle $K_{\tilde{D}}$ is trivial, with a particular trivialising section given by $(-1)^{n-1}dz_1 \wedge \ldots \wedge dz_{n-1}$. Let us denote by $(v_1, \ldots, v_n)$ the holomorphic coordinates on the total space of $K_{\tilde{D}}$ corresponding to the trivialisation, where v_k is the z_k coordinate on the base $\tilde{D}$ if $k \neq n$, and v_n is the coordinate in the fibre. More explicitly, the point $(v_1, \ldots, v_n)$ represents the vector $(-1)^{n-1}v_n\, dz_1 \wedge \ldots \wedge dz_{n-1}$ in the fibre of $K_{\tilde{D}}$ over the point $(v_1, \ldots, v_{n-1}, 0) = (z_1, \ldots, z_{n-1}, 0) \in \tilde{D}$.

In terms of the coordinates just introduced, the holomorphic covering map $\phi_\Omega : N_D\backslash\{0\} \longrightarrow K_D\backslash\{0\}$ of Corollary 2.3 determined by Ω may be written as

$$\begin{aligned} \tilde{\phi}_\Omega : N_{\tilde{D}}\backslash\{0\} &\longrightarrow K_{\tilde{D}}\backslash\{0\} \\ (w_1, \ldots, w_n) &\longmapsto (w_1, \ldots,\, w_{n-1},\, w_n^{-(\alpha-1)}) = (v_1, \ldots, v_n) \end{aligned} \tag{2.9}$$

in the local uniformising charts $(N_{\tilde{D}}, \Gamma_{N_{\tilde{D}}}, \varphi_{N_{\tilde{D}}})$ and $(K_{\tilde{D}}, \Gamma|_{\tilde{D}}, \varphi_{K_{\tilde{D}}})$ of N_D and K_D respectively. This map $\tilde{\phi}_\Omega$ is precisely the lift, to local uniformising charts, of the map that we use to pull back the Calabi ansatz cone metric and the rescaled tautological holomorphic volume form on $K_D^\times$ to $N_D\backslash\{0\}$ to obtain the Calabi-Yau cone structure $(\Omega_0,\, g_0)$ on $N_D\backslash\{0\}$.

Now, the tautological holomorphic volume form on K_D, rescaled by the factor $(1-\alpha)^{-1}$, lifts to the holomorphic volume form

$$(1-\alpha)^{-1}dv_1 \wedge \ldots \wedge dv_n$$

on the uniformising chart $(K_{\tilde{D}}, \Gamma|_{\tilde{D}}, \varphi_{K_{\tilde{D}}})$ of K_D. We therefore obtain, by virtue of (2.9), the following expression for the pullback $\tilde{\Omega}_0$ of the holomorphic volume form Ω_0 on $N_D\backslash\{0\}$ to the uniformising chart $(N_{\tilde{D}}, \Gamma_{N_{\tilde{D}}}, \varphi_{N_{\tilde{D}}})$:

$$\tilde{\Omega}_0 = (1-\alpha)^{-1}dw_1 \wedge \ldots \wedge dw_{n-1} \wedge d(w_n^{1-\alpha}) = \frac{dw_1 \wedge \ldots \wedge dw_n}{w_n^\alpha}. \tag{2.10}$$

Comparing expressions (2.8) and (2.10), it should be clear why it is necessary to work with a renormalised tautological holomorphic volume form on K_D.

The Calabi ansatz metric and some estimates. We denote by $\tilde{g}_0'$ and $\tilde{\omega}_0'$ respectively the lift to $(K_{\tilde{D}}, \Gamma|_{\tilde{D}}, \varphi_{K_{\tilde{D}}})$ of the Calabi ansatz cone metric and of its associated Kähler form on $K_D^\times$. Since the Kähler potential of $\tilde{\omega}_0'$ is $\|\cdot\|^{\frac{2}{n}}$, where $\|\cdot\|$ is the norm on $K_{\tilde{D}}$ induced from the lift to $\tilde{D}$ of the Kähler-Einstein orbifold metric on D (appropriately normalised – see Section 1.1.3), the lift $\tilde{s}$ of the radial coordinate s of the Calabi ansatz cone metric on $K_D^\times$ to $K_{\tilde{D}}$, or equivalently, the radial coordinate of $\tilde{g}_0'$, must be equal to $\|\cdot\|^{\frac{1}{n}}$. We also denote by $\tilde{g}_0$ the lift to $N_{\tilde{D}}\backslash\{0\}$ of the Calabi-Yau cone metric g_0 on $N_D\backslash\{0\}$, and by $\tilde{r}$ the lift of the corresponding radial coordinate r. It is easy to see that $\tilde{g}_0$ is a cone metric on $N_{\tilde{D}}\backslash\{0\}$ with radial coordinate $\tilde{r}$ that tends to infinity as one approaches the zero section of $N_{\tilde{D}}$. In everything that now follows, we work in the coordinates on $\tilde{U}$, $N_{\tilde{D}}$, and $K_{\tilde{D}}$ introduced above.

Let L denote the base of the Calabi-Yau cone $(N_D\backslash\{0\},\, g_0)$. Then we have conical coordinates $(r,\, x)$ on $N_D\backslash\{0\}$, where $x \in L$ and $r > 0$ is as above. Recall that in these coordinates, g_0 takes the form $g_0 = dr^2 + r^2 g_L$ for some Sasaki-Einstein metric g_L on L. For $t > 0$, we define a diffeomorphism $\nu_t : [1,\, 2] \times L \longrightarrow [t,\, 2t] \times L$ by

$$[1,\, 2] \times L \ni\, (r,\, x) \longmapsto (tr,\, x)\, \in\, [t,\, 2t] \times L.$$

Lifting this map to the uniformising chart $(N_{\tilde{D}},\, \Gamma_{N_{\tilde{D}}},\, \varphi_{N_{\tilde{D}}})$ of N_D, we obtain a map $\tilde{\nu}_t$ that necessarily takes the form

$$(w_1, \dots, w_n) \longmapsto (w_1, \dots,\, w_{n-1},\, \mu.w_n) \tag{2.11}$$

for some constant $\mu \in \mathbb{R}_+$. We next determine the value of μ. Recalling the definition of the map $\tilde{\phi}_\Omega$ from (2.9), we have

$$\begin{aligned}\tilde{r}(\tilde{\nu}_t((w_1, \dots, w_n))) &= \tilde{s}(\tilde{\phi}_\Omega(\tilde{\nu}_t((w_1, \dots, w_n))))\\ &= \tilde{s}(w_1, \dots, w_{n-1},\, \mu^{-(\alpha-1)} w_n^{-(\alpha-1)})\\ &= \mu^{-\frac{\alpha-1}{n}}\tilde{s}((w_1, \dots, w_{n-1},\, w_n^{-(\alpha-1)}))\\ &= \mu^{-\frac{\alpha-1}{n}}\tilde{s}(\tilde{\phi}_\Omega((w_1, \dots, w_n)))\\ &= \mu^{-\frac{\alpha-1}{n}}\tilde{r}((w_1, \dots, w_n)).\end{aligned} \tag{2.12}$$

In order to see how the third equality here comes about, consider a general point $(v_1, \dots, v_n) \in K_{\tilde{D}}$. This point corresponds to a vector $\mathbf{v}$ in the fibre of $K_{\tilde{D}}$ over some point $p \in \tilde{D}$. Accordingly, for $\lambda \in \mathbb{R}_+$, the point $(v_1, \dots, \lambda\, v_n)$ corresponds to the vector $\lambda\mathbf{v}$ in the fibre of $K_{\tilde{D}}$ over p. Thus, we have

$$\tilde{s}((v_1, \dots, \lambda\, v_n)) = \|\lambda\mathbf{v}\|^{\frac{1}{n}} = \lambda^{\frac{1}{n}}\|\mathbf{v}\|^{\frac{1}{n}} = \lambda^{\frac{1}{n}}\tilde{s}((v_1, \dots, v_n)),$$

which is precisely the identity we use in the third equality of (2.12).

Returning now to the above, we have just derived that $\tilde{r}(\tilde{\nu}_t((w_1, \dots, w_n))) = \mu^{-\frac{\alpha-1}{n}}\tilde{r}((w_1, \dots, w_n))$. Since $\tilde{r}(\tilde{\nu}_t((w_1, \dots, w_n))) = t\tilde{r}((w_1, \dots, w_n))$ by definition of ν_t, we deduce that $t\tilde{r}((w_1, \dots, w_n)) = \mu^{-\frac{\alpha-1}{n}}\tilde{r}((w_1, \dots, w_n))$, so that $\mu = t^{-\frac{n}{\alpha-1}}$. It follows that $\tilde{\nu}_t$ can be written as

$$\tilde{\nu}_t((w_1, \dots, w_n)) = (w_1, \dots,\, w_{n-1},\, t^{-\frac{n}{\alpha-1}}w_n)$$

in our coordinates on $N_{\tilde{D}}$.

Notice that, under the map $\tilde{\nu}_t$, we have $\tilde{\nu}_t^* w_i = w_i$ for $i \neq n$ and $\tilde{\nu}_t^* w_n = t^{-\frac{n}{\alpha-1}}w_n$. By shrinking the uniformising chart $(\tilde{U},\, \Gamma,\, \varphi)$ if necessary, it subsequently follows from Lemma 1.5 that

$$w_i = O(1) \quad \text{on } N_{\tilde{D}}\backslash\{0\} \text{ with } \tilde{g}_0\text{-derivatives if } i \neq n, \tag{2.13}$$

and that

$$w_n = O(\tilde{r}^{-\frac{n}{\alpha-1}}) \quad \text{on } N_{\tilde{D}}\backslash\{0\} \text{ with } \tilde{g}_0\text{-derivatives.} \tag{2.14}$$

Expansion of the exponential map. By the orbifold tubular neighbourhood theorem, we know that the exponential map $\exp : (T^{1,0}D)^\perp \longrightarrow X$ of the metric g is a diffeomorphism onto its image in some open neighbourhood W of the zero section of $(T^{1,0}D)^\perp$. Let $\tilde{W}$ denote the inverse image of W in the uniformising chart $((T^{1,0}\tilde{D})^\perp,\, \Gamma^\perp,\, \varphi^\perp)$ and shrink W, if necessary, so that the lift $\widetilde{\exp}$ of exp to $(T^{1,0}\tilde{D})^\perp$ satisfies $\widetilde{\exp}(\tilde{W}) \subset \tilde{U}$. This is so that we can write the $(\Gamma^\perp,\, \Gamma)$-equivariant diffeomorphism $\widetilde{\exp} : \tilde{W} \longrightarrow \widetilde{\exp}(\tilde{W}) \subset \tilde{U}$ in local holomorphic coordinates.

Recall that we have holomorphic coordinates $(z_1, \dots, z_n)$ on $\tilde{U}$ and holomorphic coordinates $(w_1,\, \dots, w_n)$ on $N_{\tilde{D}}$ with respect to which $\tilde{D} = \{z_n = 0\} = \{w_n = 0\}$. In addition, on $\tilde{D}$, we have $z_k = w_k$ for $k \neq n$. Using the metric g, we henceforth identify N_D with $(T^{1,0}D)^\perp$ and interpret the aforementioned coordinates $(w_1,\, \dots, w_n)$ on $N_{\tilde{D}}$ as coordinates on $(T^{1,0}\tilde{D})^\perp$. Thus, the coordinate $(w_1,\, \dots, w_n)$ shall now refer to the vector $w_n\frac{\partial}{\partial z_n}$ in the fibre of $(T^{1,0}\tilde{D})^\perp$ over $(w_1, \dots, w_{n-1},\, 0) = (z_1, \dots, z_{n-1},\, 0) \in \tilde{D}$. This identification also allows us to consider the holomorphic volume form Ω_0 on $N_D\backslash\{0\}$ as one on $(T^{1,0}D)^\perp\backslash\{0\}$, so that the holomorphic volume form $\tilde{\Omega}_0$ on $N_{\tilde{D}}\backslash\{0\}$ can be viewed as living on $(T^{1,0}\tilde{D})^\perp\backslash\{0\}$. The same thing can also be said for the Calabi-Yau cone metric g_0 and its associated radial function r on $N_D\backslash\{0\}$, and their respective pullbacks $\tilde{g}_0$ and $\tilde{r}$ to $N_{\tilde{D}}\backslash\{0\}$. They too can be viewed as living on the bundles $(T^{1,0}D)^\perp\backslash\{0\}$ and $(T^{1,0}\tilde{D})^\perp\backslash\{0\}$ respectively. These are viewpoints that we now henceforth assume. The estimates (2.13) and (2.14) are now re-interpreted accordingly.

With these technicalities out of the way, consider next the function

$$a : \tilde{W} \longrightarrow \mathbb{C}, \quad a(w_1, \ldots, w_n) := \widetilde{\exp}^*_{(w_1,\ldots,w_n)} z_n.$$

Clearly, by what we have just said, we must have $a(w_1, \ldots, w_{n-1}, 0) = 0$. Moreover, by choosing W so that its pre-image in each uniformising chart of $(T^{1,0}D)^{\perp}$ is a disc bundle of small radius, we may assume that the intersection of $\tilde{W}$ with each fibre of $(T^{1,0}\tilde{D})^{\perp}$ is a convex set. Thus, we may apply [32, Lemma 2.1] fibrewise, which tells us that

$$a(w_1, \ldots, w_n) = A(w_1, \bar{w}_1, \ldots, w_n, \bar{w}_n)w_n + B(w_1, \bar{w}_1, \ldots, w_n, \bar{w}_n)\bar{w}_n \tag{2.15}$$

for smooth functions A and B on $\tilde{W}$. Taking the exterior derivative of this expression yields

$$\widetilde{\exp}^*_{(w_1,\ldots,w_n)}(dz_n) = (dA)\, w_n + A\, dw_n + (dB)\, \bar{w}_n + B\, d\bar{w}_n,$$

and so, making use of the fact that $\widetilde{\exp}^*_{(w_1,\ldots,w_{n-1},\,0)}(dz_n) = dw_n$, we derive that

$$dw_n = A\, dw_n + B\, d\bar{w}_n$$

whenever $w_n = 0$. It immediately follows that

$$A(w_1, \bar{w}_1, \ldots, \bar{w}_{n-1}, 0, 0) = 1 \quad \text{and} \quad B(w_1, \bar{w}_1, \ldots, \bar{w}_{n-1}, 0, 0) = 0.$$

These last expressions allow us to apply [32, Lemma 2.1] once again to $A - 1$ and B respectively. This leads to the formulas

$$A = 1 + Cw_n + D\bar{w}_n \quad \text{and} \quad B = Ew_n + F\bar{w}_n,$$

for smooth functions C, D, E, and F on $\tilde{W}$. Combined with (2.15), we thus arrive at the expansion

$$\begin{aligned} \widetilde{\exp}^*_{(w_1,\ldots,w_n)} z_n &= (1 + Cw_n + D\bar{w}_n)w_n + (Ew_n + F\bar{w}_n)\bar{w}_n \\ &= w_n + G|w_n|^2 + Hw_n^2 + K\bar{w}_n^2 \end{aligned} \tag{2.16}$$

on $\tilde{W}$, for smooth functions G, H and K.

One can also apply the same argument to $\widetilde{\exp}^*_{(w_1,\ldots,w_n)} z_k$ for $k \neq n$. Indeed, define a function $b : \tilde{W} \longrightarrow \mathbb{C}$ by

$$b(w_1, \ldots, w_n) := \widetilde{\exp}^*_{(w_1,\ldots,w_n)} z_k - w_k.$$

Then $b(w_1, \ldots, w_{n-1}, 0) = 0$ and by [32, Lemma 2.1], we have the expansion

$$\widetilde{\exp}^*_{(w_1,\ldots,w_n)} z_k - w_k = P(w_1, \bar{w}_1, \ldots, w_n, \bar{w}_n)w_n + Q(w_1, \bar{w}_1, \ldots, w_n, \bar{w}_n)\bar{w}_n$$

on $\tilde{W}$, for smooth functions P and Q on $\tilde{W}$. Applying the exterior derivative to this expression and setting $w_n = 0$ yields the relation

$$0 = P(w_1, \bar{w}_1, \ldots, w_{n-1}, 0, 0)dw_n + Q(w_1, \bar{w}_1, \ldots, w_{n-1}, 0, 0)d\bar{w}_n,$$

where we have once again appealed to the fact that $\widetilde{\exp}^*_{(w_1,\ldots,w_{n-1},\,0)}(dz_i) = dw_i$ for all i. This last relation implies that

$$P(w_1, \bar{w}_1, \ldots, \bar{w}_{n-1}, 0, 0) = Q(w_1, \bar{w}_1, \ldots, \bar{w}_{n-1}, 0, 0) = 0,$$

and upon applying [32, Lemma 2.1] to P and Q, we find that

$$P = A_1 w_n + A_2 \bar{w}_n \quad \text{and} \quad Q = A_3 w_n + A_4 \bar{w}_n,$$

for smooth functions A_i, $i = 1, \ldots, 4$, on $\tilde{W}$. Putting all of this together, we deduce that

$$\widetilde{\exp}^*_{(w_1,\ldots,w_n)} z_k = w_k + B_1|w_n|^2 + B_2 w_n^2 + B_3 \bar{w}_n^2 \tag{2.17}$$

on $\tilde{W}$, for smooth functions B_i, $i = 1, 2, 3$, and for $k \neq n$.

2.3.1. *Some estimates involving the exponential map.* Let $F(w_1, \bar{w}_1, \ldots, w_n, \bar{w}_n)$ be a smooth function on $\tilde{W}$. Then we have the following expression for dF:

$$dF = \frac{\partial F}{\partial w_n} dw_n + \frac{\partial F}{\partial \bar{w}_n} d\bar{w}_n + \sum_{i=1}^{n-1} \frac{\partial F}{\partial w_i} dw_i + \sum_{i=1}^{n-1} \frac{\partial F}{\partial \bar{w}_i} d\bar{w}_i. \tag{2.18}$$

By shrinking W further if necessary, one can see that F and its derivatives of all orders with respect to ∂_{w_i} and $\partial_{\bar{w}_i}$ are bounded functions on $\tilde{W}$. As a consequence, from (2.18) and the estimates we have already derived on $N_{\tilde{D}} \backslash \{0\}$, we find that $|dF|_{\tilde{g}_0} = O(\tilde{r}^{-1})$ on $\tilde{W} \backslash \{0\}$. Making use of this fact in an induction argument, we deduce that

$$F = O(1) \quad \text{with } \tilde{g}_0\text{-derivatives on } \tilde{W} \backslash \{0\}. \tag{2.19}$$

Next observe that, by virtue of Lemma 1.5, we have

$$\bar{w}_n^2 / w_n = O(\tilde{r}^{-\frac{n}{\alpha-1}}) \quad \text{and} \quad w_n^2,\, \bar{w}_n^2,\, |w_n|^2 = O(\tilde{r}^{-\frac{2n}{\alpha-1}}) \quad \text{with } \tilde{g}_0\text{-derivatives on } \tilde{W} \backslash \{0\}.$$

These estimates, together with (2.14) and (2.19), now allow us to infer from (2.16) and (2.17) that on $\tilde{W} \backslash \{0\}$,

$$\frac{\widetilde{\exp}^*_{(w_1,\ldots,w_n)} z_n}{w_n} - 1 = O(\tilde{r}^{-\frac{n}{\alpha-1}}) \quad \text{with } \tilde{g}_0\text{-derivatives},$$

and that for each $k = 1, \ldots, n$,

$$\widetilde{\exp}^*_{(w_1,\ldots,w_n)}(dz_k) - dw_k = O(\tilde{r}^{-\frac{2n}{\alpha-1}-1}) \quad \text{with } \tilde{g}_0\text{-derivatives}.$$

Asymptotics of the holomorphic volume forms. We now wish to estimate the norm of the difference of the forms $\widetilde{\exp}^*(\tilde{\Omega})$ and $\tilde{\Omega}_0$ and of their derivatives of all orders on $\tilde{W} \backslash \{0\}$ with respect to $\tilde{g}_0$ and its Levi-Civita connection using what we have established thus far. Recall that the form $\tilde{\Omega}_0$ is given in coordinates $(w_1, \ldots, w_n)$ on $\tilde{W} \backslash \{0\}$ by (2.10), and that $\tilde{\Omega}$ is given in coordinates $(z_1, \ldots, z_n)$ on $\tilde{U}$ by (2.8). Using these expressions, as well as the estimates we have just derived, we find that on $\tilde{W} \backslash \{0\}$,

$$\begin{aligned}
\widetilde{\exp}^*(\tilde{\Omega}) &= \frac{\widetilde{\exp}^*(dz_1) \wedge \ldots \wedge \widetilde{\exp}^*(dz_n)}{(\widetilde{\exp}^* z_n)^\alpha} \\
&= \frac{(dw_1 + O(\tilde{r}^{-\frac{2n}{\alpha-1}-1})) \wedge \ldots \wedge (dw_n + O(\tilde{r}^{-\frac{2n}{\alpha-1}-1}))}{w_n^\alpha (1 + O(\tilde{r}^{-\frac{n}{\alpha-1}}))^\alpha} \\
&= (1 + O(\tilde{r}^{-\frac{n}{\alpha-1}}))^{-\alpha} (\Omega_0 + O(|w_n|^{-\alpha}) O(\tilde{r}^{-(n-1)} \tilde{r}^{-\frac{2n}{\alpha-1}-1})) \\
&= (1 + O(\tilde{r}^{-\frac{n}{\alpha-1}})) (\Omega_0 + O(\tilde{r}^{\frac{\alpha n}{\alpha-1}}) O(\tilde{r}^{-n-\frac{2n}{\alpha-1}})) \\
&= \tilde{\Omega}_0 + O(\tilde{r}^{-\frac{n}{\alpha-1}})
\end{aligned}$$

with $\tilde{g}_0$-derivatives. Since all the constituents of this estimate are invariant under the induced action of Γ, it descends to an estimate on $N_D|_{\varphi(\tilde{D})}$ $(\simeq (T^{1,0}D)^\perp|_{\varphi(\tilde{D})})$ and reads as:

$$\exp^*(\Omega|_{\varphi(\tilde{U} \backslash \tilde{D})}) - \Omega_0 = O(r^{-\frac{n}{\alpha-1}}) \quad \text{with } g_0\text{-derivatives}.$$

A local to global argument. In summary, we have just shown that for every $z \in D$, there exists an open neighbourhood $U(z)$ of z in the total space N_D $(\simeq (T^{1,0}D)^\perp)$ such that

$$\exp^*(\Omega) - \Omega_0 = O(r^{-\frac{n}{\alpha-1}}) \quad \text{with } g_0\text{-derivatives on } U(z) \backslash \{0\}.$$

An obvious covering argument now allows us to pass from this local statement to a global one and assert the existence of a neighbourhood U of the zero section of N_D on which

$$\exp^*(\Omega) - \Omega_0 = O(r^{-\frac{n}{\alpha-1}}) \quad \text{with } g_0\text{-derivatives on } U \backslash \{0\}.$$

This is precisely the claim of Proposition 2.4 and thus brings us to the conclusion of its proof. □

Using the previous proposition, we can now deduce from Theorem 1.11 that for each $c > 0$, there exists a unique AC Calabi-Yau metric in each μ-almost compactly supported Kähler class of $X\backslash D$ with $\mu < 0$ that is asymptotic to cg_0. We claim that, under the hypotheses of Theorem 1.15, *every* Kähler class on $X\backslash D$ is μ-almost compactly supported for some $\mu < 0$. To prove this, it suffices to show that, under the hypotheses of Theorem 1.15, $h^{0,2}(X) = 0$. For if this were the case, then an application of the Gysin sequence from Proposition A.26 would prove surjectivity of $H^{1,1}(X)$ onto the Kähler cone of $X\backslash D$, from which the claim would then follow. We shall outline the precise details of this argument in Section 2.6.

The vanishing $h^{0,2}(X) = 0$ we desire follows from the Kawamata-Viehweg vanishing theorem for Kähler orbifolds (cf. Theorem B.1). However, in order to apply this theorem, we must show that $-K_X$ is nef in the analytic sense; that is, in the sense of Definition B.2. The proof of this assertion is the subject of the next section.

2.4. **No holomorphic forms on X.** The following is the main result of interest.

Theorem 2.5. *Let X be a compact Kähler orbifold and let D be a codimension one complex suborbifold of X containing the singularities of X that, as a Baily divisor in X, satisfies $-K_X = \alpha[D]$ for some $\alpha \in \mathbb{N}$, $\alpha \geq 2$. Furthermore, assume that D has positive normal orbibundle and that the basic assumption holds for the pair (X, D). Then $h^{p,0}(X) = h^{0,p}(X) = 0$ for all $p > 0$. In particular, if X (hence D) is smooth, then X is projective algebraic.*

The key point in proving this is to establish that the line orbibundle $[D]$ is nef in the analytic sense; see Definition B.2. Once we have this, we may simply observe that

$$H^{0,p}(X) = H^p(X, \mathcal{O}_X) = H^p(X, K_X \otimes -K_X) = 0,$$

by Theorem B.1 applied to the line bundle $F = -K_X$, which is nef in the analytic sense because $[D]$ is. The fact that $\int_X c_1(F)^n > 0$ in this case follows immediately from Poincaré duality and the positivity of $[D]|_D$. Indeed, we have

$$\int_X c_1(F)^n = \alpha^n \int_X c_1([D])^n = \alpha^n \int_D c_1([D]|_D)^{n-1} > 0.$$

Notice that if X is smooth and if we already *knew* that X is projective, then $[D]$ would clearly be nef in the algebraic sense, and the vanishing would follow from the Kawamata-Viehweg vanishing theorem for projective manifolds. We will pursue this thought in Section 2.5, where we show that if X is smooth, then the projectivity of X already follows from the fact that N_D is positive.

Thus, for now, the following lemma is enough to complete the proof of Theorem 2.5.

Lemma 2.6. *Let X be a compact Kähler orbifold of complex dimension n and let D be a codimension one complex suborbifold of X containing the singularities of X with positive normal orbibundle. Furthermore, assume that for every $x \in D$, there exists a uniformising chart $(\tilde{U}, \Gamma, \varphi)$ of X with $x \in \varphi(\tilde{U})$ that admits holomorphic coordinates $(z_1, \dots, z_n)$ on $\tilde{U}$ with respect to which*

(a) *$\varphi^{-1}(D \cap \varphi(\tilde{U})) = \{z_n = 0\}$, and with respect to which*

(b) *each element $\gamma \in \Gamma$ takes the form $\begin{pmatrix} A(\gamma) & \star \\ 0 & e^{i\theta(\gamma)} \end{pmatrix}$ for some $A(\gamma) \in GL(n-1, \mathbb{C})$ and for some $\theta(\gamma) \in \mathbb{R}$, both depending on γ.*

Then the line orbibundle $[D]$ on X is nef in the analytic sense (Definition B.2).

Proof. Given a smooth positively embedded divisor in a Kähler manifold, it is shown in [24, §VIII.1] that it is always possible to find a tubular neighbourhood to which the restriction of the associated line bundle is positive. By modifying the proof of this proposition in the obvious way, and by making use of the assumption on the existence of a uniformising chart of X about each point of D satisfying conditions (a) and (b) above, one can show that the same statement also holds true

for the restriction of the line orbibundle $[D]$ to a definite tubular neighbourhood U of D in X. So let h be any Hermitian metric on $[D]$ whose curvature form in U is strictly positive and let s be a defining section of $[D]$, vanishing along D. Denote the Kähler form of X by ω. For any $\delta > 0$ and for any smooth function $f_\delta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $f_\delta(x) = x$ on $(0, \delta)$, one can define a new Hermitian metric h_δ on $[D]$ by $h_\delta(s, s) := f_\delta(h(s, s))$. This new Hermitian metric coincides with h in some open neighbourhood of D dependent on δ and its curvature form is given by

$$
\begin{aligned}
&-i\partial\bar{\partial}\log h_\delta(s,s) = \\
&\quad = i\frac{\partial h_\delta(s,s)\wedge\bar{\partial}h_\delta(s,s)}{h_\delta(s,\,s)^2} - i\frac{\partial\bar{\partial}h_\delta(s,s)}{h_\delta(s,s)} \\
&\quad = i\left(\left(\frac{f_\delta'(h(s,\,s))}{f_\delta(h(s,\,s))}\right)^2 - \frac{f_\delta''(h(s,\,s)))}{f_\delta(h(s,\,s)}\right)\partial h\wedge\bar{\partial}h - i\left(\frac{f_\delta'(h(s,\,s))}{f_\delta(h(s,\,s))}\right)\partial\bar{\partial}h(s,\,s) \\
&\quad = i\left(\left(\frac{f_\delta'(h(s,\,s))}{f_\delta(h(s,\,s))}\right)^2 - \frac{f_\delta''(h(s,\,s)))}{f_\delta(h(s,\,s)}\right)\partial h\wedge\bar{\partial}h - i\left(\frac{h\,f_\delta'(h(s,\,s))}{f_\delta(h(s,\,s))}\right)\left(\partial\bar{\partial}\log h + \frac{\partial h\wedge\bar{\partial}h}{h^2}\right) \\
&\quad = i\left(\left(\frac{f_\delta'(h(s,\,s))}{f_\delta(h(s,\,s))}\right)^2 - \frac{f_\delta''(h(s,\,s)))}{f_\delta(h(s,\,s)} - \frac{f_\delta'(h(s,\,s))}{h\,f_\delta(h(s,\,s))}\right)\partial h\wedge\bar{\partial}h + \left(\frac{h\,f_\delta'(h(s,\,s))}{f_\delta(h(s,\,s))}\right)(-i\partial\bar{\partial}\log h).
\end{aligned}
\tag{2.20}
$$

For $l > 0$, let V_l denote the connected component of the set $\{x \in X : 0 \leq h(s(x),\, s(x)) < l\}$ containing D, fix $\delta \in (0, 1)$ sufficiently small such that $V_{3\delta} \subset U$, and define a function $\tilde{g}_\delta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by $\tilde{g}_\delta(x) := \int_0^x \tilde{g}_\delta'(t)\,dt$, where

$$
\tilde{g}_\delta'(x) := \begin{cases} 1 & \text{if } x \in (0,\,\delta] \\ \cos^2(\frac{\pi}{2\delta}(x-\delta)) & \text{if } x \in (\delta,\,2\delta) \\ 0 & \text{if } x \geq 2\delta. \end{cases}
$$

One can check that $\tilde{g}_\delta \in C^2$ and that $\tilde{g}_\delta(x) = x$ on $(0, \delta)$. It is also possible to show that $\tilde{g}_\delta(x)$ satisfies the following ODI:

$$
\left(\frac{\tilde{g}_\delta'(x)}{\tilde{g}_\delta(x)}\right)^2 - \frac{\tilde{g}_\delta''(x)}{\tilde{g}_\delta(x)} - \frac{\tilde{g}_\delta'(x)}{x\tilde{g}_\delta(x)} \geq 0,
$$

with equality for $x \in (0,\, \delta) \cup (2\delta,\, +\infty)$. Now let $\varepsilon > 0$, let $k \in \mathbb{N}$, let g_k' be a smooth approximation to $\tilde{g}_\delta'$ with $g_k'(x) = \tilde{g}_\delta'(x)$ for all $x \in (0,\, \frac{\delta}{2}) \cup (\frac{5\delta}{2},\, +\infty)$ and with $\|g_k' - \tilde{g}_\delta'\|_{C^1} < \frac{1}{k}$, and set $g_k(x) := \int_0^x g_k'(t)\,dt$. (Such an approximation to $\tilde{g}_\delta'$ exists by [34, Lemma 4.1]). Then we necessarily have $\|g_k - \tilde{g}_\delta\|_{C^2} < \frac{3}{k}$ and

$$
\left(\frac{g_k'(x)}{g_k(x)}\right)^2 - \frac{g_k''(x)}{g_k(x)} - \frac{g_k'(x)}{xg_k(x)} = 0
$$

for all $x \in (0,\, \frac{\delta}{2}) \cup (\frac{5\delta}{2},\, +\infty)$. Also observe that, since the function $F : \mathbb{R}^4 \longrightarrow \mathbb{R}$ defined by

$$
F(x,\, y,\, z,\, w) := \left(\frac{z}{y}\right)^2 - \frac{w}{y} - \frac{z}{xy}
$$

is uniformly continuous on any compact subset of $\mathbb{R}^4$ for which $x,\, y \neq 0$, we have that

$$
\left(\frac{g_k'(x)}{g_k(x)}\right)^2 - \frac{g_k''(x)}{g_k(x)} - \frac{g_k'(x)}{xg_k(x)} > -\varepsilon \tag{2.21}
$$

for all $x \in [\frac{\delta}{2},\, \frac{5\delta}{2}]$, for k chosen sufficiently large. In addition, for similar reasons, we have that

$$
\frac{xg_k'(x)}{g_k(x)} > -\varepsilon \tag{2.22}
$$

for all $x \in [\frac{\delta}{2},\, \frac{5\delta}{2}]$, for k sufficiently large.

Fix k sufficiently large so that both (2.21) and (2.22) hold, and define a Hermitian metric H on $[D]$ by

$$H(s(y),\, s(y)) := \begin{cases} g_k(h(s(y),\, s(y))) & \text{for } y \in V_{3\delta} \\ g_k(3\delta) & \text{otherwise.} \end{cases}$$

Then by (2.20) and the above observations, we see that the curvature form of H satisfies $-i\partial\bar{\partial}\log H > -C\varepsilon\omega$ for some constant $C > 0$ independent of ε. That is, $[D]$ is nef in the analytic sense, as claimed. ☐

2.5. **A different approach when X is smooth.** As mentioned earlier, we will now restrict to the case where X is smooth, and provide a complementary outlook on Theorem 2.5.

Proposition 2.7. *If a compact Kähler manifold X contains a smooth divisor whose normal bundle is positive, then X is a projective algebraic variety.*

If the divisor in question is anticanonical, then not only is X almost (or weak) Fano, but one is able to appeal to the Kawamata-Viehweg vanishing theorem for projective manifolds to deduce that $h^{p,0}(X) = 0$ for all $p > 0$.

Our proof of this proposition follows along the lines of [21, Lemma 2.1].

Proof of Proposition 2.7. Due to the fact that the normal bundle N_D of D in X is positive, we know that $X \backslash D$ is 1-convex, so we may take the Remmert reduction V of $X\backslash D$. By construction, V is biholomorphic to $X\backslash D$ in a neighbourhood of infinity, and may therefore be compactified by adding the divisor D at infinity. This results in a normal compact complex space $\bar{V}$ that, in some neighbourhood of D, is smooth and biholomorphic to X.

Now D, as a divisor in $\bar{V}$, defines a line bundle $[D]$ on $\bar{V}$. The fact that D has positive normal bundle implies that the restriction $[D]|_D$ is positive. Also note that V, as a Stein space, does not contain any analytic subsets. From these observations, we deduce from [22, Satz 4, p. 347] that $[D]$ defines a positive line bundle (in the sense of Grauert [22, p. 342, Definition 2]) on $\bar{V}$. In particular, from [22, Satz 2, p. 343], we read that $\bar{V}$ admits an embedding into $\mathbb{P}^N$ for some N.

Associated to this embedding is an induced map $p : X \longrightarrow \mathbb{P}^N$ which is a biholomorphism onto its image in a neighbourhood of D. Pulling back the hyperplane line bundle on $\mathbb{P}^N$, we obtain a semi-positive line bundle (in the sense of Kodaira) on X which is positive in a neighbourhood of D. [35] now tells us that X is projective, as claimed. (It is the application of [35] that requires X to be Kähler). ☐

2.6. **Completion of the proof of Theorem 1.15.** Recall the hypotheses on X and D from Theorem 1.15.

We need to show that every Kähler class on $X\backslash D$ is μ-almost compactly supported for some $\mu < 0$. In the following lemma, we show that this is in fact true for $\mu = -2$.

Lemma 2.8. *Let X and D satisfy the hypotheses of Theorem 1.15. Then every Kähler class on $X\backslash D$ is μ-almost compactly supported with $\mu = -2$.*

We use the notation introduced in the statement of Theorem 1.15 in what follows.

Proof of Lemma 2.8. Since D is assumed to be Kähler-Einstein Fano, we see from [7, Theorem 4(i)] that $H^1(D, \mathbb{R}) = 0$. It then follows from the orbifold Gysin sequence (A.3) that the restriction map $j^* : H^2(X, \mathbb{R}) \longrightarrow H^2(X\backslash D, \mathbb{R})$ is surjective. Now, by Proposition 2.1, we know that the normal orbibundle of D in X is positive, so that, by Theorem 2.5, $h^{0,2}(X) = h^{2,0}(X) = 0$. Therefore, by Theorem A.27, we see that the restriction map $j^* : H^2(X, \mathbb{R}) \longrightarrow H^2(X\backslash D, \mathbb{R})$ actually defines a surjective map $j^* : H^{1,1}(X) \longrightarrow H^2(X\backslash D, \mathbb{R})$. The upshot of this is that, for any Kähler form ω on $X\backslash D$, we can always find a closed real (1, 1)-form σ on X such that $\omega - \sigma|_{X\backslash D} = d\gamma$ for some real one-form γ on $X\backslash D$. Using the estimates in the proof of Proposition 2.4, one can easily show

that $\exp^*(\sigma|_{X\setminus D}) = O(r^{-2})$ with g_0-derivatives. Thus, the Kähler class $[\omega]$ is -2-almost compactly supported, and the lemma is proved. $\square$

Since the complex dimension n of X satisfies $n \geq 3$, we deduce from Theorem 1.11 that in each Kähler class $[\omega]$ of $X\setminus D$ and for every $c > 0$, there exists a unique AC Calabi-Yau metric ω_c on $X\setminus D$ such that $\exp^*(\omega_c) - c\omega_0 = O(r^{\max\{-2, -\frac{n}{\alpha-1}\}+\delta})$ with g_0-derivatives, for any $\delta > 0$. This brings us to the conclusion of the proof of Theorem 1.15.

Appendix A. Orbifolds

In this appendix, we collect together all the facts from orbi-geometry that we use throughout the text. Our treatment here is taken, and indeed, in some places quoted verbatim, from [10] amongst other sources. However, Proposition A.26 does not seem to have been observed in the literature.

A.1. **Orbifolds.** We begin with the definition of an orbifold. In what follows, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

Definition A.1 (orbifold)**.** Let X be a paracompact Hausdorff space. An *orbifold chart* or a (*local*) *uniformising chart* on X is a triple $(\tilde{U},\,\Gamma,\,\varphi)$, where $\tilde{U}$ is a connected open subset of $\mathbb{F}^n$ containing the origin, Γ is a finite group acting effectively as diffeomorphisms (or biholomorphisms if $\mathbb{F}=\mathbb{C}$) of $\tilde{U}$, and $\varphi : \tilde{U} \longrightarrow U$ is a continuous map onto an open set $U \subset X$ such that $\varphi \circ \gamma = \varphi$ for all $\gamma \in \Gamma$ and the induced natural map of $\tilde{U}/\Gamma$ onto U is a homeomorphism. The finite group Γ is called a *local uniformising group*.

An *injection* or *embedding* between two orbifold charts $(\tilde{U},\,\Gamma,\,\varphi)$ and $(\tilde{U}',\,\Gamma',\,\varphi')$ is a smooth embedding $\lambda : \tilde{U} \longrightarrow \tilde{U}'$ such that $\varphi' \circ \lambda = \varphi$.

An *orbifold atlas* on X is a family $\mathcal{U} = \{(\tilde{U}_i,\,\Gamma_i,\,\varphi_i)\}_i$ of orbifold charts such that

(i) $X = \bigcup_i \varphi_i(\tilde{U}_i)$, and such that
(ii) for any two charts $(\tilde{U}_i,\,\Gamma_i,\,\varphi_i)$ and $(\tilde{U}_j,\,\Gamma_j,\,\varphi_j)$ with $U_i = \varphi_i(\tilde{U}_i)$ and $U_j = \varphi_j(\tilde{U}_j)$, and any point $x \in U_i \cap U_j$, there exists an open neighbourhood U_k of x and a chart $(\tilde{U}_k,\,\Gamma_k,\,\varphi_k)$ such that there are injections $\lambda_{ik} : (\tilde{U}_k,\,\Gamma_k,\,\varphi_k) \longrightarrow (\tilde{U}_i,\,\Gamma_i,\,\varphi_i)$ and $\lambda_{jk} : (\tilde{U}_k,\,\Gamma_k,\,\varphi_k) \longrightarrow (\tilde{U}_j,\,\Gamma_j,\,\varphi_j)$.

An atlas $\mathcal{U}$ is said to be a *refinement* of an atlas $\mathcal{V}$ if there exists an injection of every chart of $\mathcal{U}$ into some chart of $\mathcal{V}$. Two orbifold atlases are said to be *equivalent* if they have a common refinement. A smooth *orbifold* (or *V-manifold*) is a paracompact Hausdorff space X, together with an equivalence class of orbifold atlases $\mathcal{U}$. We denote such a pair by $(X, \mathcal{U})$. If every finite group Γ consists of orientation preserving diffeomorphisms and there is an atlas such that all the injections are orientation preserving diffeomorphisms, then we say that the orbifold is *orientable*.

We call an orbifold *real* or *complex* if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ in the above, respectively. The integer n appearing in the definition is called the (*real* or *complex*) *dimension* of the orbifold, whichever the case may be. We further remark that, by the slice theorem, the groups Γ_i can always be taken to be finite subgroups of $O(n,\,\mathbb{F}) \subset GL(n,\,\mathbb{F})$, where $O(n,\,\mathbb{R}) = O(n)$ is the usual orthogonal group and $O(n,\,\mathbb{C}) = U(n)$ is the unitary group. We will assume that this is always the case. Finally, note that every orbifold atlas is contained within a unique maximal atlas satisfying the required properties. Thus, with regards to this last point, we generally think of an orbifold X as a pair $(X, \mathcal{U})$, where $\mathcal{U}$ is the unique maximal atlas.

In the definition of orbifold above, the action of Γ is assumed to be effective. This condition is not always enforced in the definition of an orbifold in the literature. In order to differentiate between the two cases, we will generally refer to a pair $(X, \mathcal{U})$ as a *non-effective* orbifold if $(X, \mathcal{U})$ satisfies all of the properties of an orbifold in our sense except that the action of Γ in some local uniformising chart $(\tilde{U},\,\Gamma,\,\varphi) \in \mathcal{U}$ is non-effective.

Regarding injections, observe that, for any uniformising chart $(\tilde{U},\,\Gamma,\,\varphi)$ and for any $\gamma \in \Gamma$, γ defines an injection via $x \longmapsto \gamma \cdot x$. This is because any such γ satisfies $\varphi(\gamma \cdot x) = \gamma(x)$. We also have the following.

Lemma A.2 ([10, Lemma 4.1.2])**.** *Let $\lambda_1, \lambda_2 : (\tilde{U},\,\Gamma,\,\varphi) \longrightarrow (\tilde{U}',\,\Gamma',\,\varphi')$ be two injections. Then there exists a unique $\gamma' \in \Gamma'$ such that $\lambda_2 = \gamma' \circ \lambda_1$.*

In particular, this yields:

Corollary A.3 ([4, Corollary 9.9]). *With any injection* $\lambda : (\tilde{U}, \Gamma, \varphi) \longrightarrow (\tilde{U}', \Gamma', \varphi')$ *of uniformising charts, one may associate a group monomorphism* $\eta : \Gamma \longrightarrow \Gamma'$ *such that*

$$\lambda \circ \gamma = \eta(\gamma) \circ \lambda$$

for any $\gamma \in \Gamma$.

Now let $(X, \mathcal{U})$ be an n (real or complex) dimensional orbifold, let $x \in X$, and choose a local uniformising chart $(\tilde{U}, \Gamma, \varphi) \in \mathcal{U}$ with $x \in \varphi(\tilde{U})$. For any point $p \in \varphi^{-1}(\{x\})$, we make the following definition.

Definition A.4 (isotropy group). The *isotropy group* Γ_p of p is given by $\Gamma_p = \{\gamma \in \Gamma : \gamma(p) = p\}$.

For different points $p, q \in \varphi^{-1}(\{x\})$, the isotropy groups Γ_p and Γ_q are conjugate in $GL(n, \mathbb{F})$. Hence the conjugacy class of the isotropy group of any point in $\varphi^{-1}(\{x\})$ depends only upon x. Let us denote the conjugacy class of isotropy groups at a point $x \in X$ by Γ_x. Then we say that $x \in X$ is a *singular point* if $\Gamma_x \neq \{\mathrm{id}\}$. We denote the set of singular points of X by $\mathrm{Sing}(X)$. As it turns out, this is a closed subset of X. If it so happens that $\Gamma_x = \{\mathrm{id}\}$, then we call x a *smooth point* of X. Clearly the set of smooth points of X (that is to say, the *smooth locus* of X) is open in X. An orbifold X is a manifold if and only if $\Gamma_x = \{\mathrm{id}\}$ for all $x \in X$. For any $x \in X$, we can always find a local uniformising chart $(\tilde{U}, \Gamma, \varphi)$ with $\varphi^{-1}(\{x\}) = \{0\}$ and with Γ a finite subgroup of $GL(n, \mathbb{F})$ lying in Γ_x. (This is essentially [14, Proposition 8.2]). We call such a uniformising chart an *isotropy chart about* x.

Regarding the structure of the singular set of an orbifold, we have the following theorem.

Theorem A.5 ([4, Theorem 9.6]). *Let* $(X, \mathcal{U})$ *be a real* n*-dimensional orbifold. Then for any point* $x \in \mathrm{Sing}(X)$*, there is a uniformising chart* $(\tilde{U}, \Gamma, \varphi) \in \mathcal{U}$ *with* $x \in \varphi(\tilde{U})$ *such that* $\varphi^{-1}(\mathrm{Sing}(X) \cap \varphi(\tilde{U}))$ *is a finite union of submanifolds of* $\tilde{U}$ *of dimension* $< n$.

Next, we define the notion of a smooth map between orbifolds.

Definition A.6 (smooth map). Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be orbifolds. A map $f : X \longrightarrow Y$ is said to be *smooth* if for any point $x \in X$, there are orbifold charts $(\tilde{U}, \Gamma, \varphi)$ covering x and $(\tilde{U}', \Gamma', \varphi')$ covering $f(x)$, with the property that f maps $U = \varphi(\tilde{U})$ into $U' = \varphi'(\tilde{U}')$ and can be lifted to a smooth map $\tilde{f} : \tilde{U} \longrightarrow \tilde{U}'$ with $\varphi' \circ \tilde{f} = f \circ \varphi$.

Using this, we can define the notion of diffeomorphism of orbifolds.

Definition A.7 (diffeomorphism). Two orbifolds X and Y are *diffeomorphic* if there are smooth maps of orbifolds $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

We also have the definition of a suborbifold of an orbifold.

Definition A.8 (suborbifold). Given an orbifold X, a subset $D \subset X$ is called a *suborbifold* if for every $p \in D$, there exists a uniformising chart $(\tilde{U}, \Gamma, \varphi)$ of X with $p \in \varphi(\tilde{U})$ such that the inverse image of D in $\tilde{U}$ under φ is smooth. If X is a complex orbifold and if around each point of D, there exists a uniformising chart of X in which the pre-image of D is a complex submanifold, then we say that D is a *complex suborbifold* of X.

The most important property of a suborbifold of relevance to us is that it admits a tubular neighbourhood. This shall be discussed in Example A.22 below. Also, it should be highlighted that, all the suborbifolds we work with in this paper are implicitly assumed to be connected.

Now suppose that D is a suborbifold of an orbifold X and let $(\tilde{U}, \Gamma, \varphi)$ be a uniformising chart of X with $\tilde{D} := \varphi^{-1}(D \cap \varphi(\tilde{U}))$ non-empty, smooth and connected. Then, since the chart $(\tilde{D}, \Gamma, \varphi|_{\tilde{D}})$ is a uniformising chart on D, and since the collection of such uniformising charts, together with the corresponding injections induced from X, constitute an orbifold atlas on D, we see that a (complex) suborbifold is itself a (complex) orbifold, although possibly a non-effective one. Note that the

converse of this statement need not be true: a subvariety of X which is an orbifold need not be a suborbifold, the reason being that the singularities of a suborbifold are intimately related to those of the ambient space.

A.2. **Orbibundles.** The notions of a vector bundle and a principal bundle also have a counterpart in the orbifold category.

Definition A.9 (orbibundle)**.** An *orbibundle* (or *V-bundle*) over an orbifold $(X, \mathcal{U})$ consists of a fibre bundle $B_{\tilde{U}}$ over $\tilde{U}$ for each uniformising chart $(\tilde{U}, \Gamma, \varphi) \in \mathcal{U}$ with Lie group G and fibre F a smooth G-manifold which is independent of $\tilde{U}$, together with a homomorphism $h_{\tilde{U}} : \Gamma \longrightarrow G$ satisfying the following.

(i) If b lies in the fiber over $\tilde{x} \in \tilde{U}$, then for each $\gamma \in \Gamma$, $h_{\tilde{U}}(\gamma)b$ lies in the fibre over $\gamma^{-1}\tilde{x}$.
(ii) If the map $\lambda : (\tilde{U}, \Gamma, \varphi) \longrightarrow (\tilde{U}', \Gamma', \varphi')$ is an injection, then there is a bundle map $\lambda^* : B_{\tilde{U}'}|_{\lambda(\tilde{U})} \longrightarrow B_{\tilde{U}}$ satisfying the condition that if $\gamma \in \Gamma$, and γ' is the unique element of Γ' such that $\lambda \circ \gamma = \gamma' \circ \lambda$, then $h_{\tilde{U}}(\gamma) \circ \lambda^* = \lambda^* \circ h_{\tilde{U}'}(\gamma')$, and if $\lambda' : (\tilde{U}', \Gamma', \varphi') \longrightarrow (\tilde{U}'', \Gamma'', \varphi'')$ is another such injection, then $(\lambda \circ \lambda')^* = \lambda'^* \circ \lambda^*$.

If the fibre F is a (respectively complex) vector space of dimension r and G acts on F as linear transformations, then the orbibundle is called a (resp. *complex*) *vector* orbibundle of *rank r*. Similarly, if F is the Lie group G with its left action, then the orbibundle is called a *principal orbibundle*.

By choosing the local uniformising charts of X small enough, one can always take $B_{\tilde{U}}$ to be the product $\tilde{U} \times F$, in which case we say that the local uniformising system is *fine*. An orbibundle with a fine system of uniformising charts is determined by the following [3].

Lemma A.10. *Assuming the notation used in Definition A.9, an orbibundle B over X with a fine uniformising system is characterised precisely by the following data:*

$$h_{\tilde{U}}(\gamma)(x, u) = (\gamma^{-1}x, \eta_{\tilde{U}}(\gamma)(x) \cdot u),$$
$$\lambda^*(\lambda(x), u) = (x, \xi_\lambda(\lambda(x)) \cdot u),$$

where $(x, u) \in U \times F$, and $\eta_{\tilde{U}}(\gamma) : U \longrightarrow G$ and $\xi_\lambda : \lambda(U) \longrightarrow G$ are holomorphic maps (or smooth maps in the real case) that satisfy the following conditions:

(i) $\eta_{\tilde{U}}(\gamma_2)(\gamma_1^{-1}x)\eta_{\tilde{U}}(\gamma_1)(x) = \eta_{\tilde{U}}(\gamma_1\gamma_2)(x)$.
(ii) $\eta_{\tilde{U}}(\gamma)(z)\xi_\lambda(\lambda(x)) = \xi_\lambda(\lambda(\gamma^{-1}x))\eta_{\tilde{U}'}(\gamma')(\lambda(x))$.
(iii) $\xi_\lambda(\lambda(x))\xi_{\lambda'}(\lambda' \circ \lambda(x)) = \xi_{\lambda' \circ \lambda}(\lambda' \circ \lambda(x))$.

The case of particular interest to us here is when B is a complex vector orbibundle of rank one and both $\eta_{\tilde{U}}(\gamma)$ and $\xi_{\tilde{U}}$ are non-zero holomorphic functions, in which case we say that B is a (*holomorphic*) *line orbibundle*. Note that in some cases, a vector orbibundle of rank one over an orbifold may actually be a genuine line bundle. When this happens, we are able to take $h_{\tilde{U}} = \mathrm{id}$ for all local uniformising neighbourhoods $\tilde{U}$ and we say that the line orbibundle is *absolute*. For example, the trivial line orbibundle $X \times \mathbb{C}$ is absolute.

The *total space* E of an orbibundle over an orbifold X is itself a (generally non-effective) orbifold. Assuming that the orbifold atlas $\{(\tilde{U}_i, \Gamma_i, \varphi_i)\}_i$ of X is fine with respect to E, we can take as an orbifold atlas for E the collection $\{(B_{\tilde{U}_i}, \Gamma_i^*, \varphi_i^*)\}_i$, where, for each i, $\tilde{U}_i \times F \simeq B_{\tilde{U}_i}$, where the action of the local uniformising group Γ_i^* is the natural extension of that defined by Γ_i, and where φ_i^* is the projection onto the quotient $B_{\tilde{U}_i}/\Gamma_i^*$. Explicitly, for each i, an element $\gamma \in \Gamma_i^* \simeq \Gamma_i$ acts by sending a point $(\tilde{x}_i, b) \in \tilde{U}_i \times F$ to the point $(\gamma^{-1}\tilde{x}_i, h_{\tilde{U}_i}(\gamma)b)$. Associated to this orbifold atlas of E, we have, for each i, the following commutative diagram:

$$\begin{tikzcd} B_{\tilde{U}_i} \arrow[r, "\tilde{\pi}"] \arrow[d, "\varphi_i^*"'] & \tilde{U}_i \arrow[d, "\varphi_i"] \\ U_i \times F \arrow[r, "\pi"] & U_i, \end{tikzcd}$$

where $\tilde{\pi}$ and π are the obvious projections. Finally, the injections of this orbifold atlas on the total space of E are derived from those of the orbifold atlas on X in the following sense. Each injection $\widehat{\lambda}_{ji} : B_{\tilde{U}_i} \longrightarrow B_{\tilde{U}_j}$ between local uniformising charts $(B_{\tilde{U}_i}, \Gamma_i^*, \varphi_i^*)$ and $(B_{\tilde{U}_j}, \Gamma_j^*, \varphi_j^*)$ of the total space of E assumes the form $\widehat{\lambda}_{ji}(p, q) = (\lambda_{ji}(p), g_{\lambda_{ji}}(p)(q)))$ for some injection $\lambda_{ji} : (\tilde{U}_i, \Gamma_i, \varphi_i) \longrightarrow (\tilde{U}_j, \Gamma_j, \varphi_j)$ and for some smooth map $g_{\lambda_{ji}} : \tilde{U}_i \longrightarrow F$. Here, for each k, we understand the splitting $B_{\tilde{U}_k} \simeq \tilde{U}_k \times F$.

Now we ask under what conditions is the total space of a principal orbibundle smooth.

Lemma A.11 ([10, Lemma 4.2.8]). *Let G be a Lie group and P the total space of a principal orbibundle over an orbifold $(X, \mathcal{U})$. Then P is a smooth manifold if and only if the maps $h_{\tilde{U}_i}$ are injective for all i.*

As with smooth fibre bundles, we have the notion of sections of bundles in the orbifold category.

Definition A.12. Let E be an orbibundle over an orbifold X. Then a *section* σ of E over the open set $W \subset X$ is a section $\sigma_{\tilde{U}_i}$ of the bundle $B_{\tilde{U}_i}$ for each orbifold chart $(\tilde{U}_i, \Gamma_i, \varphi_i)$ of X such that $U_i \subset W$ and such that for any $\tilde{x}_i \in \tilde{U}_i$, the following conditions are satisfied.

(i) For each $\gamma \in \Gamma_i$, $\sigma_{\tilde{U}_i}(\gamma^{-1}\tilde{x}_i) = h_{\tilde{U}_i}(\gamma)\sigma_{\tilde{U}_i}(\tilde{x}_i)$.
(ii) If $\lambda_{ji} : (\tilde{U}_i, \Gamma_i, \varphi_i) \longrightarrow (\tilde{U}_j, \Gamma_j, \varphi_j)$ is an injection, then $\lambda_{ji}^*(\sigma_{\tilde{U}_j}(\lambda_{ji}(\tilde{x}_i))) = \sigma_{\tilde{U}_i}(\tilde{x}_i)$.

If each of the local sections $\sigma_{\tilde{U}_i}$ is continuous, smooth, holomorphic, etc., then we say that σ is *continuous, smooth, holomorphic,* etc., respectively.

So a section σ of E can be viewed locally as a Γ-invariant section of $B_{\tilde{U}}$ over $\tilde{U}$ for each uniformising chart $(\tilde{U}, \Gamma, \varphi)$ of X. On the other hand, given a local section $\sigma_{\tilde{U}}$ of $B_{\tilde{U}}$ over a uniformising chart $(\tilde{U}, \Gamma, \varphi)$, we can always construct a Γ-invariant local section by "averaging over the group", i.e., we define an invariant local section $\sigma^I_{\tilde{U}}$ of $B_{\tilde{U}}$ by

$$\sigma^I_{\tilde{U}} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sigma_{\tilde{U}} \circ \gamma.$$

Notice that this determines a well-defined map from the underlying space, namely $\sigma_U : U = \varphi(\tilde{U}) \longrightarrow B_{\tilde{U}}$.

The notion of an isomorphism between fibre bundles also extends to the orbifold category.

Definition A.13. Let B^1 and B^2 be two orbibundles over an orbifold $(X, \mathcal{U})$ and suppose that over each orbifold chart $(\tilde{U}, \Gamma, \varphi) \in \mathcal{U}$, B^i is defined by a fibre bundle $B^i_{\tilde{U}}$ with fibre a smooth G-manifold for some Lie group G, together with a homomorphism $h^i_{\tilde{U}} : \Gamma \longrightarrow G$, for $i = 1, 2$. Then we say that B^1 *is isomorphic to* B^2 if the following two conditions are satisfied.

(i) For each uniformising chart $(\tilde{U}, \Gamma, \varphi) \in \mathcal{U}$, there exists a fibre bundle isomorphism $\theta_{\tilde{U}} : B^1_{\tilde{U}} \longrightarrow B^2_{\tilde{U}}$ that is equivariant with respect to the action of Γ on $B^1_{\tilde{U}}$ and $B^2_{\tilde{U}}$ induced by the homomorphisms $h^1_{\tilde{U}}$ and $h^2_{\tilde{U}}$ respectively.
(ii) If $\lambda : (\tilde{U}, \Gamma, \varphi) \longrightarrow (\tilde{U}', \Gamma', \varphi')$ is an injection between uniformising charts of X, then the following diagram is commutative:

$$\begin{tikzcd} B^1_{\tilde{U}'}|_{\lambda(\tilde{U})} \arrow[r, "\lambda^*"] \arrow[d, "\theta_{\tilde{U}'}"'] & B^1_{\tilde{U}} \arrow[d, "\theta_{\tilde{U}}"] \\ B^2_{\tilde{U}'}|_{\lambda(\tilde{U})} \arrow[r, "\lambda^*"'] & B^2_{\tilde{U}} \end{tikzcd}$$

Here, λ^* is as in Definition A.9.

The final concept that we need to introduce regarding orbibundles is that of a metric on a vector orbibundle.

Definition A.14 (metric)**.** Let $(X, \mathcal{U})$ be an orbifold and let B be a vector orbibundle over X such that $B_{\tilde{U}}$ is the product bundle over each $\tilde{U} \in \mathcal{U}$. A *metric* a for B is a collection of functions $\{a_{\tilde{U}}\}_{\tilde{U} \in \mathcal{U}}$, where, for each local uniformising chart $(\tilde{U}, \Gamma, \varphi) \in \mathcal{U}$, $a_{\tilde{U}}$ is a function that assigns to each $x \in \tilde{U}$ a (symmetric or Hermitian) positive definite bilinear form $a_{\tilde{U}}(x)$ in the fibre of $B_{\tilde{U}}$ over x such that the following conditions are satisfied.

(i) If Y and Z belong to the fibre of $B_{\tilde{U}}$ over $x \in \tilde{U}$, then

$$a_{\tilde{U}}(x)(Y, Z) = a_{\tilde{U}}(\gamma^{-1} \cdot x)(h_{\tilde{U}}(\gamma)Y, h_{\tilde{U}}(\gamma)Z)$$

for $\gamma \in \Gamma$.

(ii) If $\lambda : (\tilde{U}, \Gamma, \varphi) \longrightarrow (\tilde{U}', \Gamma', \varphi')$ is an injection, then

$$a_{\tilde{U}}(x)(\lambda^* Y, \lambda^* Z) = a_{\tilde{U}'}(\lambda(x))(Y, Z)$$

for Y and Z in the fibre of $B_{\tilde{U}'}$ over $\lambda(x)$.

(iii) If ψ and σ are smooth sections of B, then the function $a(\psi, \sigma)$ on X defined by

$$a(\psi, \sigma)(\varphi(x)) = a_{\tilde{U}}(x)(\psi_{\tilde{U}}(x), \sigma_{\tilde{U}}(x)), \quad x \in \tilde{U},$$

is smooth.

We remark that, via a partition of unity argument [2], it is possible to show that:

Proposition A.15. *Every vector orbibundle admits a metric.*

We close this section with a discussion of the most natural examples of orbibundles associated to an orbifold.

Example A.16. The most trivial example of a vector orbibundle on an orbifold $(X, \mathcal{U})$ is the trivial line bundle $X \times \mathbb{R}$. Here we have $G = GL(1, \mathbb{R})$ and we take $B_{\tilde{U}} = \tilde{U} \times \mathbb{R}$ for each uniformising chart $(\tilde{U}, \Gamma, \varphi) \in \mathcal{U}$. The definitions of both $h_{\tilde{U}}$ and λ^*_{ji} should be clear in this case.

Smooth sections of this bundle are called *smooth functions* on X and the set of smooth sections is denoted by $C^\infty(X, \mathbb{R})$. Simply put, this set consists of real-valued smooth functions on the smooth locus of X whose pullbacks extend to real-valued smooth functions on each uniformising chart.

Example A.17. The tangent orbibundle TX of a real n-dimensional orbifold $(X, \mathcal{U})$, the analogue in the orbifold category of the tangent bundle of a manifold, is constructed as follows.

For each uniformising chart $(\tilde{U}_i, \Gamma_i, \varphi_i) \in \mathcal{U}$, we take $B_{\tilde{U}_i} = T\tilde{U}_i$, the tangent bundle of the open manifold $\tilde{U}_i \subset \mathbb{R}^n$. We take $G = GL(n, \mathbb{R})$ and for each $\gamma \in \Gamma_i$, we define $h_{\tilde{U}_i}(\gamma) = (df_\gamma)^{-1}$, where $f_\gamma(x) = \gamma \cdot x$ for each $x \in \tilde{U}_i$. Moreover, given an injection $\lambda_{ji} : (\tilde{U}_i, \Gamma_i, \varphi_i) \longrightarrow (\tilde{U}_j, \Gamma_j, \varphi_j)$, we take as the corresponding bundle map λ^*_{ji} the inverse of the mapping of tangent vectors induced by λ_{ji}.

The orbifold atlas $\mathcal{U}^*$ induced on the total space of TX by the above construction has charts of the form $(\tilde{U}_i \times \mathbb{R}^n, \Gamma^*_i, \varphi^*_i)$, where the local uniformising group Γ^*_i is Γ_i acting on $\tilde{U}_i \times \mathbb{R}^n$ via $(\tilde{x}, v) \longmapsto (\gamma^{-1} \cdot \tilde{x}, h_{\tilde{U}_i}(\gamma)v)$, and $\varphi^*_i : \tilde{U}_i \times \mathbb{R}^n \longrightarrow (\tilde{U}_i \times \mathbb{R}^n)/\Gamma^*_i$ is the natural quotient projection. Here, it is assumed that the choice of orbifold atlas $\mathcal{U}$ for X is fine for TX.

The *tangent space* T_xX to a point $x \in X$ is the "fibre" of the tangent orbibundle TX of X at x. It may be identified with $T_{\tilde{x}}\tilde{U}/\Gamma$, where $(\tilde{U}, \Gamma, \varphi)$ is an isotropy chart about x, $\tilde{x}$ is the point of $\tilde{U}$ that is mapped to x under φ, and the action of Γ on $T_{\tilde{x}}\tilde{U}$ is that defined by $h_{\tilde{U}}$ above. At smooth points x of X, it can be seen to coincide with the usual notion of tangent space.

We call a smooth section of TX a *vector field* on X.

To construct the cotangent orbibundle, or more generally, to construct the dual B' of any vector orbibundle B, we take as the fibres of B' the dual of the fibres of B, with defining maps $h'_{\tilde{U}}(\gamma)$ and $\lambda^{*\prime}$

the inverses of the transposes of the corresponding maps for B. For tensor products of orbibundles, we take the tensor product of the corresponding fibres and maps. Together, these two constructions allow us to build, for $k = 0, 1, \ldots$, the *kth exterior power* Λ^k of the cotangent orbibundle of an orbifold, thus allowing us to say what is meant by a smooth differential k-form on an orbifold.

Definition A.18. Let X be an orbifold. A *smooth differential k-form α* on X is a global section of the orbibundle Λ^k. Equivalently, α is a smooth k-form on the smooth locus of X whose pullback $\varphi^*\alpha$ to each local uniformising chart $(\tilde{U}, \Gamma, \varphi)$ of X extends to a smooth differential k-form on $\tilde{U}$.

By considering the relevant properties in the smooth case, one can also say what is meant by a Riemannian metric, a complex structure, etc. on an orbifold. For example, we have the following.

Definition A.19. A *Riemannian metric g* on an orbifold X is a Riemannian metric g_i on each local uniformising chart $(\tilde{U}_i, \Gamma_i, \varphi_i)$ that is invariant under the local uniformising group Γ_i, and such that the injections $\lambda_{ji} : (\tilde{U}_i, \Gamma_i, \varphi_i) \longrightarrow (\tilde{U}_j, \Gamma_j, \varphi_j)$ are isometries; that is, $\lambda_{ji}^*(g_j|_{\lambda_{ji}(U_i)}) = g_i$.

An orbifold with a Riemannian metric is called a *Riemannian orbifold*.

This definition is of course equivalent to Definition A.14 with $B = TX$, in which case Proposition A.15 reads as:

Proposition A.20. *Every orbifold admits a Riemannian metric.*

Now, a *volume form* on an orientable Riemannian orbifold (X, g) is a nowhere-vanishing, top-degree form on X whose pullback to each uniformising chart defines a volume form for the metric defining g in the uniformising chart in consideration. As is the case for orientable Riemannian manifolds, a volume form is defined up to sign, with the choice of sign consistent with the orientation.

As well as the tangent orbibundle, we also have the following definition of the canonical and anticanonical orbibundle of a complex orbifold.

Example A.21. Recalling Lemma A.10, let f be a holomorphic map from an open subset $U \subset \mathbb{C}^n$ into $\mathbb{C}^n$, write f as $(f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$ in local holomorphic coordinates z_i on $\mathbb{C}^n$, and set

$$J(f)(z) = [\det(\partial f_i/\partial z_j)]_z^{-1} \qquad \text{for } z \in U.$$

Then, with the notation as in the aforementioned lemma, the *canonical orbibundle* K_X of a complex orbifold X is determined precisely by the following data:

$$\eta_{\tilde{U}}(\gamma)(z) = J(f_{\gamma^{-1}})(z) \quad \text{and} \quad \xi_\lambda(\lambda(z)) = J(\lambda^{-1})(\lambda(z)).$$

Here, as before, $f_{\gamma^{-1}}(z) = \gamma^{-1} \cdot z$ for $z \in \tilde{U}$. Over a uniformising chart $(\tilde{U}_i, \Gamma_i, \varphi_i)$ of X, the vector bundle $B_{\tilde{U}_i}$ that defines K_X is the canonical bundle $K_{\tilde{U}_i}$ of the open manifold $\tilde{U}_i \subset \mathbb{C}^n$.

The *anticanonical orbibundle* $-K_X$ of X is the line orbibundle dual to K_X.

For a suborbifold, we are able to give a well-defined notion of a normal orbibundle.

Example A.22. Let D be a suborbifold of an orbifold X and let TD and TX denote the tangent orbibundle of D and X respectively. The *normal orbibundle* N_D of D in X is the orbibundle on D defined as the cokernel of the natural injection $TD \subset TX|_D$. By definition, it fits into the following short exact sequence:

$$0 \longrightarrow TD \longrightarrow TX|_D \longrightarrow N_D \longrightarrow 0.$$

For general subvarieties of an orbifold, a normal orbibundle need not exist, since an arbitrary subvariety may not even admit a tangent orbibundle. Thus, the fact that a suborbifold is itself an orbifold is one of the reasons why its normal orbibundle can be defined. Using a Riemannian metric, it is possible to identify the normal orbibundle of a suborbifold $D \subset X$ with a sub-orbibundle of $TX|_D$. (We have not explicitly defined the term "sub-orbibundle" here, but its definition should be clear). Indeed, the map $(TD)^\perp \ni v \longmapsto [v] \in N_D$, where $(TD)^\perp$ is the g-orthogonal complement of TD in TX, yields such an identification.

Now let $(\tilde{U}, \Gamma, \varphi)$ be a uniformising chart of a Riemannian orbifold (X, g) with the inverse image $\tilde{D} := \varphi^{-1}(D \cap \varphi(\tilde{U}))$ of D in $\tilde{U}$ non-empty, smooth, and connected. Then, as a final note, we remark that the vector bundle that defines the vector orbibundle $(TD)^{\perp}$ on the induced uniformising chart $(\tilde{D}, \Gamma, \varphi|_{\tilde{D}})$ of D is $(T\tilde{D})^{\perp}$, the orthogonal complement, with respect to $\varphi^* g$, of $T\tilde{D}$ in $T\tilde{U}|_{\tilde{D}}$. The induced action of Γ on $(T\tilde{D})^{\perp}$ here is that given by the restriction of the induced action of Γ on $T\tilde{U}|_{\tilde{D}}$.

The orbibundle dual to N_D also has a name: it is called the *conormal orbibundle* N_D^* *of* D *in* X. Unlike the normal orbibundle, which is not a sub-orbibundle of anything, the conormal orbibundle is a sub-orbibundle of $T^*X|_D$; for if $\tilde{D}$ denotes the preimage of D in some uniformising chart $(\tilde{U}, \Gamma, \varphi)$ of X, and $\tilde{D}$ is smooth and connected, then the vector bundle that defines N_D^* over the induced chart $(\tilde{D}, \Gamma, \varphi|_{\tilde{D}})$ of D is $\{(x, \xi) \in T^*\tilde{U}|_{\tilde{D}} : x \in \tilde{D}, \quad \xi \in T_x^*\tilde{U}, \quad \text{and} \quad \xi|_{T_x\tilde{D}} = 0\}$. (This is just the conormal bundle $N_{\tilde{D}}^*$ of $\tilde{D}$ in $\tilde{U}$). If we furthermore assume that $\tilde{D}$ is of real codimension k and that it is cut out by k real-valued smooth functions f_i, $i = 1, \ldots, k$, then the covectors $df_i|_{\tilde{D}}$, $i = 1, \ldots, k$, actually serve as a spanning set for $N_{\tilde{D}}^*$.

Now, the metric g associated to a Riemannian orbifold (X, g) determines, at each point $x \in X$, an exponential map $\exp_x^g : T_xX \longrightarrow X$. To define this map, we choose an isotropy chart $(\tilde{U}, \Gamma, \varphi)$ about x and set $\tilde{x} = \varphi^{-1}(x) \in \tilde{U}$. Then, with $\mathrm{pr} : T_{\tilde{x}}\tilde{U} \longrightarrow T_xX \simeq T_{\tilde{x}}\tilde{U}/\Gamma$ denoting the quotient map with respect to the induced linear action of Γ on $T_{\tilde{x}}\tilde{U}$, and with $\exp_{\tilde{x}}^{\tilde{g}} : T_{\tilde{x}}\tilde{U} \longrightarrow \tilde{U}$ denoting the exponential map of the Γ-invariant metric $\tilde{g} = \varphi^* g$ on $\tilde{U}$ at the point $\tilde{x}$, we choose $v \in T_xX$ with g-norm equal to one and $\tilde{v} \in \mathrm{pr}^{-1}(\{v\})$, and define $\exp_x^g(tv)$, $t \in \mathbb{R}$, to be equal to $\varphi \circ \exp_{\tilde{x}}^{\tilde{g}}(t\tilde{v})$ for small t and to be an extension by the projection of the geodesic on local uniformising charts for large t. Γ-equivariance guarantees that $\exp_x^g$ is well-defined and gives rise to the following commutative diagram:

$$\begin{tikzcd} T_{\tilde{x}}\tilde{U} \arrow[r, "\exp_{\tilde{x}}^{\tilde{g}}"] \arrow[d, "\mathrm{pr}"'] & \tilde{U} \arrow[d, "\varphi"] \\ T_xX \cong T_{\tilde{x}}\tilde{U}/\Gamma \arrow[r, "\exp_x^g"] & U \cong \tilde{U}/\Gamma. \end{tikzcd}$$

As with smooth submanifolds, there exists a neighbourhood of the zero section of the normal orbibundle N_D of a suborbifold D of an orbifold X on which the exponential map is a diffeomorphism onto its image. This is the orbifold version of the so-called tubular neighbourhood theorem. For a discussion, see [28, Remark 2.1].

A.3. **Complex orbifolds.** In this section, we concentrate solely on complex orbifolds. They come equipped with a Γ-invariant tensor field J of type $(1, 1)$ which describes the complex structure on the tangent orbibundle. This gives rise to the holomorphic tangent orbibundle $T^{1,0}X$, the anti-holomorphic tangent orbibundle $T^{0,1}X$, and the orbibundles $\Lambda^{p,q}$ of differential forms of type (p, q) in the usual way. We are also able to define the standard notions of Hermitian and Kähler metrics, in analogy with Definition A.19, and to extend the definition of the normal (respectively conormal) orbibundle of a suborbifold in the obvious way to that of a *holomorphic normal* (resp. *conormal*) *orbibundle* of a complex suborbifold. Our notation for these latter-mentioned orbibundles shall remain the same as for their real counterparts. In analogy with Proposition A.20, we have that every complex orbifold admits a Hermitian metric.

Now, given an n-dimensional complex orbifold X, we can endow it with the structure of a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ as follows. At a point $x \in X$, the stalk of the structure sheaf $\mathcal{O}_X$ is isomorphic to the local ring $\mathcal{O}_{\mathbb{C}^n}^{\Gamma}$ of germs of Γ-invariant holomorphic functions on $\mathbb{C}^n$, where Γ is the local uniformising group of a uniformising chart $(\tilde{U}, \Gamma, \varphi)$ covering x. In particular, with this definition, we see that the stalk of $\mathcal{O}_X$ at a smooth point of X is isomorphic to the ring $\mathcal{O}_{\mathbb{C}^n}$ of convergent power series in $\mathbb{C}^n$. We also note the following properties of the $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$.

Proposition A.23 ([10, Proposition 4.4.4]). *The locally ringed space $(X, \mathcal{O}_X)$ associated to a complex orbifold has the following properties.*

(i) *$(X, \mathcal{O}_X)$ is a reduced normal complex space.*

(ii) *The set of singular points of the underlying complex space of X is a reduced complex subspace of X and has complex codimension at least two in X.*

(iii) *The smooth points of the underlying complex space of X is a complex manifold and a dense open subset of X.*

Thus, by Hartog's theorem, any holomorphic function on the complement of the singular set of X extends to a holomorphic function on the whole of X, as long as the singular set is compact.

The correct notion of a divisor on a complex orbifold is that of a "Baily divisor". On each local uniformising chart $(\tilde{U}, \Gamma, \varphi)$, we consider a Cartier divisor $D_{\tilde{U}}$ on $\tilde{U}$. Since $\tilde{U}$ is an open subset of affine space, $D_{\tilde{U}}$ corresponds to an invertible sheaf $\mathcal{D}_{\tilde{U}}$ on $\tilde{U}$. In the following, we denote by $\mathcal{D}_{\tilde{U},x}$ the stalk of this sheaf at a point $x \in \tilde{U}$.

Definition A.24. An *orbidivisor* or *Baily divisor* on a complex orbifold X is a Cartier divisor $D_{\tilde{U}}$ on each local uniformising chart $(\tilde{U}, \Gamma, \varphi)$ of X that satisfies the following two conditions.

(i) If, for each $x \in \tilde{U}$ and $\gamma \in \Gamma$, f belongs to the module $\mathcal{D}_{\tilde{U},\gamma\cdot x}$ over the ring of local holomorphic functions at $\gamma \cdot x \in \tilde{U}$ which are locally multiples of $\mathcal{D}_{\tilde{U}}$, then $f \circ \gamma$ belongs to $\mathcal{D}_{\tilde{U},x}$.

(ii) If $\lambda : (\tilde{U}, \Gamma, \varphi) \longrightarrow (\tilde{U}', \Gamma', \varphi')$ is an injection and if f belongs to $\mathcal{D}_{\tilde{U}',\lambda(y)}$ at $\lambda(y) \in \lambda(\tilde{U})$, then $f \circ \gamma$ belongs to $\mathcal{D}_{\tilde{U},y}$.

A Baily divisor is called *absolute* if on each local uniformising system $(\tilde{U}, \Gamma, \varphi)$, the divisor $D_{\tilde{U}}$ can be written as $D_{\tilde{U}} = (f_{\tilde{U}})$, where $f_{\tilde{U}}$ is the quotient of Γ-invariant holomorphic functions on $\tilde{U}$.

In the same way that divisors on a complex manifold induce complex lines bundles, Baily divisors on an orbifold induce complex line orbibundles. In particular, a codimension one complex suborbifold D inside a compact complex orbifold $(X, \mathcal{U})$ naturally defines a Baily divisor in X, and so gives rise to a complex line orbibundle $[D]$ on X. To see this explicitly, suppose that $\mathcal{V} := \{(\tilde{U}_i, \Gamma_i, \varphi_i)\}_i \subset \mathcal{U}$ is a covering of X by uniformising charts, where, for each i, the inverse image $\tilde{D}_i := \varphi_i^{-1}(\varphi_i(\tilde{U}_i) \cap D)$ is cut out by some holomorphic function f_i on $\tilde{U}_i$ with $df_i|_{\tilde{D}_i} \neq 0$. (If $\tilde{D}_i$ is empty for some i, then we set $f_i \equiv 1$). Then, assuming the notation of Lemma A.10, the complex line orbibundle $[D]$ is determined by the following data:

$$\eta_{\tilde{U}_k}(\gamma)(z) = (f_{\tilde{U}_k} \circ \gamma^{-1})(z)/f_{\tilde{U}_k}(z), \quad \text{where } (\tilde{U}_k, \Gamma_k, \varphi_k) \in \mathcal{V} \text{ and } \gamma \in \Gamma_k,$$

and

$$\xi_\lambda(\lambda(z)) = f_{\tilde{U}_k}(z)/(f_{\tilde{U}_l} \circ \lambda)(z),$$

where $(\tilde{U}_i, \Gamma_i, \varphi_i) \in \mathcal{V}$ for $i = k, l$, $\lambda : (\tilde{U}_k, \Gamma_k, \varphi_k) \longrightarrow (\tilde{U}_l, \Gamma_l, \varphi_l)$ is an injection, and $z \in \tilde{U}_k$. We also get for free a defining section σ of this orbibundle that vanishes to order one along D. In the uniformising chart $(\tilde{U}_i, \Gamma_i, \varphi_i)$, it is given simply by $\sigma|_{\tilde{U}_i} = \sigma_{\tilde{U}_i} = f_i$.

In the case that a Baily divisor is absolute, we actually get an induced absolute complex line orbibundle, that is, an induced complex line bundle.

A.4. **Cohomology of orbifolds.** Let X be an orbifold. Recall from above that a smooth differential k-form α on X is a smooth differential k-form on the smooth locus of X such that, for each uniformising chart $(\tilde{U}, \Gamma, \varphi)$ of X, $\varphi^*\alpha$ extends to a smooth differential k-form on $\tilde{U}$. If X is complex, then an analogous statement holds for (p, q)-forms on X. Let us denote the sheaf of differential k-forms on X by A^k, and for X complex, the sheaf of (p, q)-forms by $A^{p,q}$. Then, with the obvious definition of the exterior derivative d on X, we have the complex

$$0 \longrightarrow A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \cdots \tag{A.1}$$

and the associated *de Rham cohomology groups*

$$H^p(X,\,\mathbb{R}) := \frac{\ker(d : A^p(X) \longrightarrow A^{p+1}(X))}{\operatorname{im}(d : A^{p-1}(X) \longrightarrow A^p(X))}.$$

We also have the *compactly supported de Rham cohomology groups* $H^*_c(X,\,\mathbb{R})$, defined as the cohomology of the subcomplex of (A.1) comprising of forms with compact support. For orbifolds with a finite "good" cover, both of these groups turn out to be finite dimensional. In particular, compact orbifolds admit a finite good cover, hence have finite dimensional de Rham and compactly supported de Rham cohomology groups. The proof of this assertion follows along the same lines as in the smooth case [8, p.43 & p.44]. One first proves a Mayer-Vietoris sequence for de Rham and compactly supported de Rham cohomology on orbifolds using the fact that an orbifold admits a partition of unity subordinate to any locally finite open cover [4, Theorem B.12]. Then, using the fact that every orbifold admits a "good" cover [33, Corollary 1.2.5], one proceeds by induction on the number of open sets in the good cover.

An important property of the de Rham cohomology of orbifolds is that it coincides with the simplicial cohomology of the underlying topological space of the orbifold [37]. In particular, it follows that de Rham cohomology of orbifolds is a homotopy invariant, as it is for smooth manifolds.

In the case that X is complex, we have in addition to the above the usual $\bar\partial$-operator and the *Dolbeault cohomology groups*

$$H^{p,\,q}(X) := \frac{\ker(\bar\partial : A^{p,\,q}(X) \longrightarrow A^{p,\,q+1}(X))}{\operatorname{im}(\bar\partial : A^{p,\,q-1}(X) \longrightarrow A^{p,\,q}(X))}$$

associated to the complex $(A^{p,\,*}(X),\,\bar\partial)$. As is the case for complex manifolds, there exists a relationship between these cohomology groups and the sheaf cohomology $H^*(X,\,\Omega^p)$ of the sheaf Ω^p of germs of holomorphic p-forms on X. Namely,

$$H^{p,\,q}(X) = H^q(X,\,\Omega^p). \tag{A.2}$$

This is Dolbeault's theorem in the category of complex orbifolds [3]. If X is in addition compact, then each space $H^{p,\,q}(X)$ is finite dimensional. This follows from Baily's generalisation [2] of harmonic theory to the orbifold category. We call the numbers $h^{p,\,q}(X) := \dim_{\mathbb{C}} H^{p,\,q}(X)$ the *Hodge numbers* of X.

Integration of a smooth compactly supported top-degree form α on an orbifold X is defined in a similar manner as for smooth manifolds. Namely, we take a partition of unity $\{\rho_i\}_i$ subordinate to a locally finite covering of X by uniformising charts $\{(\tilde U_i,\,\Gamma_i,\,\varphi_i)\}_i$ with $\operatorname{supp}\rho_i \subset \varphi_i(\tilde U_i)$, and we define

$$\int_X \alpha := \sum_i \frac{1}{|\Gamma_i|}\int_{\tilde U_i} \varphi_i^*(\rho_i\alpha).$$

With this definition, all of the standard integration techniques, such as Stokes' theorem, hold on a compact orbifold. We also have Poincaré duality.

Theorem A.25 ([6, Theorem 1.8])**.** *Let X be an orbifold, $n = \dim_{\mathbb{R}} X$, and $k \in \mathbb{N}\cup\{0\}$ with $k < n$. Then the pairing $H^k(X,\,\mathbb{R}) \times H^{n-k}_c(X,\,\mathbb{R}) \longrightarrow \mathbb{R}$ given by $([\mu],[\eta]) \longmapsto \int_X \mu\wedge\eta$ is well-defined and non-degenerate.*

This theorem allows us to identify $H^k(X,\,\mathbb{R})$ with $(H^{n-k}_c(X,\,\mathbb{R}))^*$, the dual of $H^{n-k}_c(X,\,\mathbb{R})$, via $[\mu] \longmapsto \int_X \mu \wedge \cdot$.

Next let X be a compact complex manifold and let D be a smooth divisor in X. Then there exists a long exact sequence of cohomology, called the Gysin sequence, relating the de Rham cohomology of X and D to that of $X\backslash D$ – see for example [26, §II4.1]. A version of this sequence, in terms of de Rham cohomology as defined above, also exists in the orbifold category. Indeed, we have:

Proposition A.26 (Gysin sequence for orbifolds)**.** *Let X be a compact complex orbifold of complex dimension n and let D be a codimension one complex suborbifold of X containing the singularities*

of X. Then there is a long exact sequence of de Rham cohomology:

$$\cdots \longrightarrow H^p(X,\,\mathbb{R}) \xrightarrow{j^*} H^p(X\backslash D,\,\mathbb{R}) \longrightarrow H^{p-1}(D,\,\mathbb{R}) \longrightarrow H^{p+1}(X,\,\mathbb{R}) \longrightarrow \cdots \tag{A.3}$$

Here, $j : X\backslash D \longrightarrow X$ denotes the inclusion of $X\backslash D$ in X.

Proof. Let U be a tubular neighbourhood of D in X and define $A_0^p(U,\,\partial U)$ to be the subspace of $A^p(U)$ comprising of p-forms that vanish to all orders at the boundary ∂U of U, so that an element α lies in $A_0^p(U,\,\partial U)$ if it is the restriction to U of some element $\tilde{\alpha} \in A^p(V)$ that vanishes on $V\backslash U$ for some open neighbourhood V of $\overline{U}$ in X. Then we have a short exact sequence of complexes

$$0 \longrightarrow (A_0^*(U,\,\partial U),\,d) \longrightarrow (A^*(X),\,d) \longrightarrow (A^*(X\backslash U),\,d) \longrightarrow 0,$$

which gives rise to the following long exact sequence of de Rham cohomology:

$$\ldots \longrightarrow H_0^p(U,\,\partial U) \longrightarrow H^p(X,\,\mathbb{R}) \longrightarrow H^p(X\backslash U,\,\mathbb{R}) \longrightarrow H_c^{p+1}(U) \longrightarrow \ldots \tag{A.4}$$

Here,

$$H_0^p(U,\,\partial U) := \frac{\ker(d : A_0^p(U,\,\partial U) \longrightarrow A_0^{p+1}(U,\,\partial U))}{\operatorname{im}(d : A_0^{p-1}(U,\,\partial U) \longrightarrow A_0^p(U,\,\partial U)}$$

is the *p-th relative de Rham cohomology group of the pair* $(U,\,\partial U)$.

Now, since de Rham cohomology is a homotopy invariant, we know that the groups $H^p(X\backslash U,\,\mathbb{R})$ and $H^p(X\backslash D,\,\mathbb{R})$ coincide. Also, it is possible to show that $H_0^p(U,\,\partial U) = H_c^p(U,\,\mathbb{R})$. And finally, by diffeomorphism invariance, we know that $H_c^p(U,\,\mathbb{R}) = H_c^p(W,\,\mathbb{R}) = H_c^p(N_D,\,\mathbb{R})$ for some neighbourhood W of the zero section of the normal orbibundle N_D of D in X. After making the appropriate substitutions in (A.4), we thus see that this aforementioned long exact sequence can be rewritten as:

$$\ldots \longrightarrow H_c^p(N_D,\,\mathbb{R}) \longrightarrow H^p(X,\,\mathbb{R}) \longrightarrow H^p(X\backslash D,\,\mathbb{R}) \longrightarrow H_c^{p+1}(N_D,\,\mathbb{R}) \longrightarrow \ldots \tag{A.5}$$

Next observe that

$$\begin{aligned} H_c^p(N_D,\,\mathbb{R}) &= (H^{2n-p}(N_D,\,\mathbb{R}))^* \quad \text{by Poincaré duality on } N_D \text{ and the fact that both} \\ &\qquad\qquad H_c^p(N_D,\,\mathbb{R}) \text{ and } H^p(N_D,\,\mathbb{R}) \text{ are finite dimensional,} \\ &= (H^{2n-p}(D,\,\mathbb{R}))^* \quad \text{by homotopy invariance of de Rham cohomology,} \\ &= H_c^{p-2}(D,\,\mathbb{R}) \quad \text{by Poincaré duality on } D, \\ &= H^{p-2}(D,\,\mathbb{R}) \quad \text{by compactness of } D. \end{aligned}$$

And so, after replacing $H_c^p(N_D,\,\mathbb{R})$ in (A.5) with $H^{p-2}(D,\,\mathbb{R})$, we arrive at the long exact sequence (A.3). □

A.5. **Kähler and Fano orbifolds.** Now we focus our attention on specifically Kähler orbifolds. First recall that a *Kähler orbifold* is a complex Hermitian orbifold with Hermitian metric g and complex structure J such that the corresponding Γ-invariant Hermitian 2-form $\omega(X,\,Y) = g(JX,\,Y)$ defined in each uniformising chart $(\tilde{U},\,\Gamma,\,\varphi)$ is closed. The closed form ω is, as suspected, called the *Kähler form* and the corresponding cohomology class $[\omega] \in H^2(X,\,\mathbb{R})$ the *Kähler class.* Now Kähler orbifolds are not unlike Kähler manifolds in that many of the properties of Kähler manifolds carry over to the orbifold case. For example, the $i\partial\bar{\partial}$-lemma holds on compact Kähler orbifolds, as well as the following version of the Hodge decomposition theorem.

Theorem A.27 ([9, p. 8])**.** *Let X be a compact Kähler orbifold. Then the following direct sum decomposition holds:*

$$H^r(X,\,\mathbb{R}) \otimes \mathbb{C} = \sum_{p+q=r} H^{p,\,q}(X).$$

Moreover, with respect to complex conjugation on $H^r(X,\,\mathbb{R}) \otimes \mathbb{C}$, one has

$$\overline{H^{p,\,q}}(X) = H^{q,\,p}(X).$$

In particular, we have the equality $h^{p,q}(X) = h^{q,p}(X)$.

The proof of this theorem, and others like it, follow the usual proofs verbatim by using (where necessary) Baily's generalisation [2] of harmonic theory to the orbifold category.

Next consider a complex line orbibundle B over a complex orbifold $(X, \mathcal{U})$ of complex dimension n, endowed with a Hermitian metric a. For each uniformising chart $(\tilde{U}, \Gamma, \varphi) \in \mathcal{U}$, $a_{\tilde{U}}$ is, by definition, a positive real-valued function on $\tilde{U}$, and by part (i) of Definition A.14, it is easy to see that the associated purely complex (1, 1)-form $\phi_{\tilde{U}} = \bar{\partial}\partial \log a_{\tilde{U}}$ on $\tilde{U}$ is Γ-invariant and that the collection of differential forms $\{\phi_{\tilde{U}}\}$ define a global purely complex closed differential form ϕ on X. We call this form ϕ the *curvature form* of the Hermitian metric a, and the cohomology class in $H^2(X, \mathbb{R})$ defined by the corresponding real (1, 1)-form $\frac{i}{2\pi}\phi$ on X the *first Chern class* $c_1(B)$ of the orbibundle B. It is not hard to show that the class $c_1(B)$ is independent of the choice of Hermitian metric on B. Moreover, one can show that it has the following two additional properties in common with the corresponding notion on complex manifolds.

(i) It is additive with respect to the tensor product of complex line orbibundles.

(ii) If B is induced by a Baily divisor D, then $c_1(B)$ is the "Poincaré dual" of D, i.e., $c_1(B)$ satisfies

$$\int_D \eta = \int_X c_1(B) \wedge \eta \qquad \text{for every } \eta \in A^{2n-2}(X).$$

Finally, if a Hermitian metric can be chosen on B whose curvature form is positive definite, or equivalently, if $c_1(B)$ can be represented by a Kähler form, then we say that B is *positive.* By the Kodaira-Baily Embedding theorem [3], any compact complex orbifold admitting a positive holomorphic line orbibundle is necessarily projective-algebraic. One reason for introducing the above concepts is that they allow us to extend the definition of a Fano manifold to the orbifold category. Indeed, we have:

Definition A.28. A compact complex orbifold X is *Fano* if the anticanonical orbibundle $-K_X$ is positive, or equivalently, if the cohomology class $-c_1(K_X) \in H^2(X, \mathbb{R})$ can be represented by a Kähler form on X.

Thus, by positivity of their anticanonical orbibundle, every Fano orbifold is necessarily projective-algebraic.

Now let X be a compact Kähler orbifold. Then a canonical representative of the cohomology class $-c_1(K_X)$ is the *Ricci form* ρ associated to any Kähler metric ω on X. Here, ρ is defined just as it is on smooth Kähler manifolds, that is, as $\frac{i}{2\pi}$ times the curvature of the Hermitian metric induced on the orbibundle $-K_X$ by ω. If ρ is a positive form on X, then it is clear from Definition A.28 that X is Fano. If, in addition, both ρ and ω fulfil the requirement that their respective pullbacks $\tilde{\rho}$ and $\tilde{\omega}$ to each local uniformising chart of X satisfy $\tilde{\rho} = \lambda.\tilde{\omega}$ for some $\lambda > 0$, then we say that X is a *Kähler-Einstein Fano orbifold.* This concept is important in the Calabi ansatz construction of Calabi-Yau cones – see Section 1.1.3.

We conclude this appendix by noting that Yau's proof [31] for solutions of the complex Monge-Ampère equation on compact Kähler manifolds also extends to the orbifold category.

Theorem A.29 (cf. [12, Theorem 4.3])**.** *Let X be a compact Kähler orbifold of complex dimension n with Kähler form ω. Then, for every $f \in C^\infty(X, \mathbb{R})$, there exists a smooth function $\phi \in C^\infty(X, \mathbb{R})$, unique up to constant, such that*

$$(\omega + i\partial\bar{\partial}\phi)^n = Ae^f \omega^n.$$

Here, $A = \int_X \omega^n / \int_X e^f \omega^n$.

Appendix B. Kawamata-Viehweg for Kähler orbifolds

B.1. **Overview.** Let X be a compact Kähler orbifold and let E be any holomorphic line orbibundle on X. Then we have a complex

$$0 \longrightarrow A^{0,0}(E, X) \xrightarrow{\bar{\partial}} A^{0,1}(E, X) \xrightarrow{\bar{\partial}} A^{0,2}(E, X) \xrightarrow{\bar{\partial}} \cdots,$$

where $A^{p,q}(E, X)$ denotes global sections of the sheaf $A^{p,q}(E)$ of E-valued (p, q)-forms on X, and the $\bar{\partial}$-operator is extended in the obvious way. We also have the cohomology groups $H^*(X, E)$ associated to this complex, which, for $E = \mathcal{O}_X$, are nothing more than the Dolbeault cohomology groups $H^{0,*}(X)$. Our aim in this appendix is to prove the following theorem regarding the vanishing of these groups in the special case that $E = K_X \otimes F$ for a holomorphic line orbibundle F on X.

Theorem B.1. *Let X be a compact Kähler orbifold of complex dimension n and let F be a holomorphic line orbibundle on X. Suppose that F is nef and that $\int_X c_1(F)^n > 0$. Then $H^q(X, K_X \otimes F) = 0$ for all $q > 0$.*

Some explanation here is required regarding the meaning of the term "nef" in the statement of this theorem. We have the following definition.

Definition B.2. Let X be a compact Kähler orbifold of complex dimension n with Kähler form ω and let F be a holomorphic line orbibundle over X. Then we say that F is *nef* (*in the analytic sense*) if, for any $\varepsilon > 0$, the first Chern class $c_1(F) \in H^2(X, \mathbb{R})$ of F can be represented by a closed real $(1, 1)$-form γ_ε such that $\varepsilon\omega + \gamma_\varepsilon$ is positive definite.

It is easy to see that if F is nef and if $\int_X c_1(F)^n \neq 0$, then necessarily $\int_X c_1(F)^n > 0$.

In the case that X is projective and F is a holomorphic line bundle over X, nefness of F in the analytic sense is equivalent to nefness of F in the algebraic sense. Furthermore, when X is projective and $F = [D]$ for a smooth divisor D in X is nef, bigness of F is equivalent to the condition $\int_X c_1([D])^n > 0$. Theorem B.1 may therefore be considered an analogue of the Kawamata-Viehweg vanishing theorem [5, Corollary 15.20] for compact Kähler orbifolds. Our proof of Theorem B.1 is essentially an "orbifolded" version of a special case of the proof of an analogous result by Enoki [20, Theorem 0.1] for compact Kähler manifolds, by which we mean we follow Enoki's proof with $\nu(F) =$ "numerical Kodaira dimension of F" $= n$ verbatim and replace any results he appeals to with the corresponding result in the orbifold category. Note that, as well as Enoki's result and the above, there do exist other analogues, indeed generalisations, of the Kawamata-Viehweg vanishing theorem in the complex analytic setting – see for example [19], although it should be emphasised that Theorem B.1 does not follow as a corollary of that paper.

Let us now present the proof of Theorem B.1. We begin by introducing the set-up and notation. Our notation will follow that in [20] as closely as possible.

B.2. **Set-up and notation.** Let (X, g) be a Riemannian orbifold with Riemannian metric g. Then we have a notion for what it means for a sequence of differential forms on X to "converge as a current".

Definition B.3. Let X be an orbifold of real dimension n and let $\{\sigma_j\}$ be a sequence of differential k-forms on X. Then we say that σ_j *converges as a current* on X as $j \longrightarrow \infty$ if for every compactly supported $(n-k)$-form μ on X, the sequence $\{\int_X \sigma_j \wedge \mu\}$ converges as $j \longrightarrow \infty$.

Now assume that X is compact and as before, let $A^p(X)$ denote the set of p-forms on X. Then we have the L^2-inner product $(\cdot\,,\,\cdot)_{L^2}$ on $A^p(X)$, defined by

$$(\alpha, \beta)_{L^2} := \int_X g(\alpha, \beta)\, d\mathrm{vol}_g \quad \text{for } \alpha, \beta \in A^p(X).$$

Here, $d\mathrm{vol}_g$ denotes the volume form associated to g.

If X is moreover Kähler with Kähler metric g, then we have $n!\, d\mathrm{vol}_g = \omega^n$, where n is the complex dimension of X and ω is the Kähler form associated to g. Henceforth assuming that X is compact and Kähler, consider a holomorphic line orbibundle F over X. An F-valued $(p,\,q)$-form on X is just a smooth global section of the orbibundle $\Lambda^{p,\,q} \otimes F$ over X, or equivalently, a global section of the sheaf $A^{p,\,q}(F)$. We denote the space of such forms by $A^{p,\,q}(F,\,X)$. Given a Hermitian metric h on F, we obtain, in the obvious way, a Hermitian metric on the orbibundle $\Lambda^{p,\,q} \otimes F$ induced by g and h. Let us denote this induced Hermitian metric by $\langle\cdot,\,\cdot\rangle$. Then we have an induced Hermitian L^2-inner product, which we denote by $\langle\cdot,\,\cdot\rangle_{L^2}$, on the space $A^{p,\,q}(F,\,X)$. Explicitly, this is given by

$$\langle\alpha,\,\beta\rangle_{L^2} := \int_X \langle\alpha,\,\beta\rangle\, d\mathrm{vol}_g \quad \text{for } \alpha,\,\beta \in A^{p,\,q}(F,\,X).$$

As on a Kähler manifold, we have the Lefschetz operator $L : A^{p,\,q}(F,\,X) \longrightarrow A^{p+1,\,q+1}(F,\,X)$ defined by multiplication by the Kähler form ω. We let Λ denote the adjoint of this operator with respect to $\langle\cdot,\,\cdot\rangle_{L^2}$. More generally, for any form α on X, we have an operator $l(\alpha)$ defined by $l(\alpha)\xi = \alpha \wedge \xi$ for $\xi \in A^{p,\,q}(F,\,X)$. Hence we have $L = l(\omega)$.

Finally, we say what it means for an element of $A^{p,\,q}(X,\,F)$ to be harmonic. Since F is a holomorphic line orbibundle, we have an extension of the $\bar\partial$-operator to elements of $A^{p,q}(X,\,F)$. This operator has a formal adjoint $\bar\partial^*$ with respect to the inner product $\langle\cdot,\,\cdot\rangle_{L^2}$. Using these operators, we define the *$\bar\partial$-Laplacian* $\Box_{\bar\partial}$ acting on $A^{p,\,q}(X,\,F)$ by $\Box_{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. We then say that an element of $A^{p,\,q}(X,\,F)$ is *harmonic* if it lies in the kernel of $\Box_{\bar\partial}$. Note that it is possible to write the operator $\Box_{\bar\partial}$ in terms of the $(1,\,0)$-component of the *Chern connection* with respect to h on F (cf. [20, eqn. (1.1)]), but as we shall not use this expression explicitly, we omit the details.

We next state and prove some preliminary results.

B.3. Preliminary results.

Proposition B.4 (cf. [20, Proposition 3.1]). *Let X be a compact Kähler orbifold of complex dimension n, let F be a holomorphic line orbibundle over X, and let γ be the curvature form of a Hermitian metric on F. Moreover, let $q \geq 0$. Then, for any F-valued harmonic $(n,\,q)$-form ξ on X, we have*

$$i\langle l(\gamma + \partial\bar\partial f)\Lambda\xi,\, e^f\xi\rangle_{L^2} \leq 0$$

for any smooth real-valued function f on X.

Proof. The proof of this result proceeds exactly as in the proof of [20, Proposition 3.1] so we omit the details. □

Lemma B.5 (cf. [20, Lemma 3.4]). *Let X be a compact Riemannian orbifold with Riemannian metric g, and let Δ_g and $d\,\mathrm{vol}_g$ denote the Laplacian and volume form of g respectively. Moreover, let $\{f_j\}$ be a sequence of smooth functions on X with $\int_X f_j\, d\,\mathrm{vol}_g = 0$ and suppose that Δf_j converges as a current on X as $j \longrightarrow \infty$. Then f_j also converges as a current on X as $j \longrightarrow \infty$.*

Proof. For a smooth test function φ, set $\varphi = c + \varphi_0$ where $c = \frac{1}{\mathrm{vol}_g(X)}\int_X \varphi\, d\mathrm{vol}_g$. (Here, $\mathrm{vol}_g(X) = \int_X d\mathrm{vol}_g$). Then $\langle f_j,\,\varphi\rangle_{L^2} = \langle f_j,\,\varphi_0\rangle_{L^2}$, since $\int_X f_j\, d\mathrm{vol}_g = 0$. Now, by [15], we know that there exists a smooth function ψ on X with $\Delta_g\psi = \varphi_0$. Thus, we find that

$$\langle f_j,\,\varphi\rangle_{L^2} = \langle f_j,\,\Delta_g\psi\rangle_{L^2} = \langle\Delta_g f_j,\,\psi\rangle_{L^2} \longrightarrow \text{const.} \qquad \text{as } j \longrightarrow \infty.$$

That is, f_j converges as a current as $j \longrightarrow \infty$, as claimed. □

Lemma B.6 (cf. [20, Lemma 3.5]). *Let $\{f_j\}$ be a sequence of smooth functions on a compact Riemannian orbifold $(X,\,g)$ that are uniformly bounded above, and use the notation as in Lemma B.5. Suppose that $\{f_j\}$ converges as a current on X. Then $\lim_{j\longrightarrow\infty}\int_U e^{f_j} h\, d\mathrm{vol}_g \neq 0$ for any open subset U of X and for any strictly positive function h on U.*

Proof. Suppose that $\lim_{j\longrightarrow\infty}\int_U e^{f_j} h\, d\mathrm{vol}_g = 0$ for some open subset U of X and for some strictly positive function h on U. Take a relatively compact open subset U_0 of U whose closure is contained

within U and a smooth non-negative function ψ supported on U with $\psi|_{U_0} = 1$. (Note that it is always possible to construct such a function ψ by [4, Corollary B.13]). Set $V_j := \{p \in U_0 : e^{f_j(p)} < \varepsilon_j\}$, where

$$\varepsilon_j := \int_{U_0} e^{f_j} h \, d\mathrm{vol}_g / \int_{V_j} h \, d\mathrm{vol}_g.$$

Then we have that $0 \leq \varepsilon_j . \mathbb{1}_{U_0 \setminus V_j} . h \leq e^{f_j} h$, where $\mathbb{1}_E$ denotes the indicator function of a set E. From this, it follows that $\int_{U_0} h \, d\mathrm{vol}_g \leq 2 \int_{V_j} h \, d\mathrm{vol}_g$, so that $\varepsilon_j \longrightarrow 0$ as $j \longrightarrow \infty$. Treating ψh as a function on the whole of X in the obvious way and setting $M := \sup_{j,x} f_j(x)$, we therefore find that

$$\begin{aligned}\int_X (\psi h f_j) \, d\mathrm{vol}_g = M \int_X (\psi h) \, d\mathrm{vol}_g + \int_X \psi h (f_j - M) \, d\mathrm{vol}_g \leq \text{const.} + \int_{V_j} \psi h (f_j - M) \, d\mathrm{vol}_g \\ \leq \text{const.} + (\log(\varepsilon_j) - M) \int_{V_j} h \, d\mathrm{vol}_g \longrightarrow -\infty \quad \text{as } j \longrightarrow \infty.\end{aligned}$$

This is a clear contradiction to the assumption that $\{f_j\}$ converges as a current. □

We are now in a position to prove Theorem B.1.

B.4. **Proof of Theorem B.1.** (cf. [20, pp. 65-67]). We fix, once and for all, a Hermitian metric h on F and a Kähler form ω on X. We denote by $\langle \cdot \, , \, \cdot \rangle$ the inner product on sections of the orbibundle $K_X \otimes F$ induced by ω and h, and we let ξ denote the unique harmonic representative of $H^q(X, K_X \otimes F)$ relative to h and ω, which exists by virtue of [30, Proposition 2.2]. Clearly ξ defines an F-valued harmonic (n, q)-form on X relative to h and ω. Supposing that $q > 0$ and $\xi \neq 0$, we shall derive a contradiction.

Define

$$p(\varepsilon) := \int_X (\varepsilon\omega + i\gamma)^n / \int_X \omega^n,$$

where γ is the curvature form of (F, h). Since F is nef and $i\gamma$ represents $c_1(F)$, we can find, by making use of the $i\partial\bar{\partial}$-lemma, a smooth real-valued function ψ_ε such that $\varepsilon\omega + i\gamma + i\partial\bar{\partial}\psi_\varepsilon$ is a Kähler form on X. By Theorem A.29, we know that there exists a smooth real-valued function φ_ε on X such that $\varepsilon\omega + i\gamma + i\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon)$ is a Kähler form on X with

$$(\varepsilon\omega + i\gamma + i\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon))^n = p(\varepsilon)\omega^n. \tag{B.1}$$

Let $f_\varepsilon = \psi_\varepsilon + \varphi_\varepsilon$ and set $\rho_\varepsilon = \varepsilon\omega + i(\gamma + \partial\bar{\partial} f_\varepsilon)$. Then

$$\begin{aligned}\int_X \rho_\varepsilon \wedge \omega^{n-1} &= \varepsilon \int_X \omega^n + \int_X i\gamma \wedge \omega^{n-1} + \int_X i\partial\bar{\partial} f_\varepsilon \wedge \omega^{n-1} \\ &= \varepsilon \int_X \omega^n + \int_X i\gamma \wedge \omega^{n-1} + \frac{1}{2n} \underbrace{\int_X (\Delta_\omega f_\varepsilon)\, \omega^n}_{=0} \xrightarrow{\varepsilon \to 0} \int_X i\gamma \wedge \omega^n,\end{aligned}$$

so that $\int_X \rho_\varepsilon \wedge \omega^{n-1}$ is bounded as $\varepsilon \longrightarrow 0$. Since ρ_ε is a Kähler form, it follows that for any open subset U of X and for any smooth bump function η supported on U with $0 \leq \eta \leq 1$, $\int_U \eta . \rho_\varepsilon \wedge \omega^{n-1}$ is bounded as $\varepsilon \longrightarrow 0$.

Next let $x \in X$, let $(\tilde{U}, \Gamma, \varphi)$ be a local isotropy chart about x, and let $V \subset \varphi(\tilde{U})$ be an open subset containing x. Also, choose a smooth bump function η on X, supported on $\varphi(\tilde{U})$, with $\overline{V} \subset \{y \in X : \eta(y) = 1\}$ and $0 \leq \eta \leq 1$. (Recall that it is always possible to construct such a function by [4, Corollary B.13]). By the above observations, it is clear that $\int_K \tilde{\rho}_\varepsilon \wedge \tilde{\omega}^{n-1}$ is bounded as $\varepsilon \longrightarrow 0$ for every compact subset K of $\varphi^{-1}(V)$. Here, $\tilde{\rho}_\varepsilon$ and $\tilde{\omega}$ denotes the pullback, via φ, of ρ_ε and ω respectively to $\tilde{U}$. From [38, Lemma 2.11], we thus see that there exists a sequence $\{\varepsilon_j\}$ with $\lim_{j \longrightarrow \infty} \varepsilon_j = 0$ such that $\tilde{\rho}_{\varepsilon_j}$ converges as a current on V as $j \longrightarrow \infty$. By choosing a partition of unity subordinate to a finite open cover of X by uniformising charts (which we can do by [4, Theorem B.12]), and by implementing the usual technique of diagonal subsequence selection, it is easy to see that this argument extends to show the existence of a subsequence, also denoted by $\{\varepsilon_j\}$,

with $\lim_{j\longrightarrow\infty}\varepsilon_j = 0$, such that ρ_{ε_j} converges as a current on X as $j \longrightarrow \infty$. It now follows from Lemma B.5 that $\{f_{\varepsilon_j}\}$ converges as a current on X under the normalisation $\int_X f_{\varepsilon_j} = 0$.

As well as this convergence, we also have the bound $i\partial\bar\partial f_{\varepsilon_j} > -c\omega$ for some constant c independent of ε_j. This just follows from the fact that ρ_{ε_j} is positive definite. Using Green's formula for the Laplacian of a compact Riemannian orbifold (which can be derived from [15]), one can now show, via a standard argument, that $\sup_X f_{\varepsilon_j} \leq C$ for some constant C independent of ε_j. To simplify notation, we now set $\varepsilon = \varepsilon_j$ in what follows, so that $f_\varepsilon = f_{\varepsilon_j}$ and $\rho_\varepsilon = \rho_{\varepsilon_j}$. Moreover, when we let $j \longrightarrow \infty$ (and $\varepsilon_j \longrightarrow 0$), we now simply write $\varepsilon \longrightarrow 0$.

With $(\tilde U, \Gamma, \varphi)$ and x (as well as our other notation) as above, choose a local unitary frame $(\theta_1, \ldots, \theta_n)$ of $\Lambda^{1,0}\tilde U$ with respect to $\tilde\omega$ that diagonalises $\tilde\rho_\varepsilon$ at the point $\tilde x := \varphi^{-1}(x)$. Let us write $\tilde\rho_\varepsilon = i\sum_j \lambda_j\theta_j \wedge \bar\theta_j$ at $\tilde x$. Notice that each λ_j is positive. From (B.1), we see that $\prod_{i=1}^n \lambda_i = p(\varepsilon)$ at $\tilde x$, since on $\tilde U$, this equation reads as $\tilde\rho_\varepsilon^n = p(\varepsilon)\tilde\omega^n$. It is also clear that $p(\varepsilon)$ is a polynomial in ε of degree n with non-negative coefficients. Thus, we may write $p(\varepsilon) = \sum_{j=0}^n a_j\varepsilon^j$ for some constants $a_j \geq 0$. Recalling the fact that $\int_X c_1(F)^n > 0$, we must actually have $a_0 > 0$. As a result, we find that

$$0 < a_0 \leq \sum_{j\geq 0} a_j\varepsilon^j = p(\varepsilon) = \prod_{j=1}^n \lambda_j \tag{B.2}$$

at $\tilde x$, where $a_0 > 0$ is independent of ε.

Next consider the non-negative continuous function $u_\varepsilon := \frac{\langle l(\rho_\varepsilon)\Lambda\xi,\, \xi\rangle}{\langle \xi,\, \xi\rangle}$ on X, and let

$$U(\varepsilon) := \{x \in X : u_\varepsilon(x) < 2\varepsilon q\}.$$

We claim that there exists a constant $C_0 > 0$ independent of ε such that $\int_{U(\varepsilon)}\omega^n \geq C_0$. Indeed, suppose the contrary; that is, suppose that $\int_{U(\varepsilon)}\omega^n \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Then, by Proposition B.4, we have the bound $\int_X u_\varepsilon\, dv_\varepsilon \leq \varepsilon q \int_X dv_\varepsilon$, where $dv_\varepsilon := \langle\xi,\, \xi\rangle e^{f_\varepsilon}\omega^n$. (Since ξ is of type $(n,\, q)$, we have that $\omega \wedge \Lambda\xi = q\xi$). It thus follows from Chebychev's inequality that $\int_{U(\varepsilon)} dv_\varepsilon \geq \frac{1}{2}\int_X dv_\varepsilon$. After noting that $\sup_X\langle\xi,\, \xi\rangle e^{f_\varepsilon}$ is bounded as $\varepsilon \longrightarrow 0$, we use this last inequality to derive that

$$\frac{1}{2}\int_X \langle\xi,\, \xi\rangle e^{f_\varepsilon}\omega^n \leq \int_{U(\varepsilon)}\langle\xi,\, \xi\rangle e^{f_\varepsilon}\omega^n \leq M\int_{U(\varepsilon)}\omega^n \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

where M is a constant independent of ε such that $\langle\xi,\, \xi\rangle e^{f_\varepsilon} \leq M$ on X. On the other hand, since ξ is harmonic, we know from Aronszajn's unique continuation theorem (see for example [13, p. 6]) that the pullback $\varphi^*\xi$ of ξ to any local uniformising chart $(\tilde U, \Gamma, \varphi)$ of X does not vanish identically on $\tilde U$. Hence there is an isotropy chart $(\tilde U', \Gamma', \varphi')$ about some point of X on which the pullback of ξ never vanishes. The above inequality then implies that $\int_{\varphi'(\tilde U')}\langle\xi,\, \xi\rangle e^{f_\varepsilon}\omega^n \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Since $\sup_X f_\varepsilon$ is bounded above as $\varepsilon \longrightarrow 0$, this contradicts Lemma B.6.

Let $\mathrm{tr}_\omega\,\rho_\varepsilon$ denote the trace of ρ_ε with respect to ω, i.e., set $\mathrm{tr}_\omega\,\rho_\varepsilon := \sum_{i=1}^{2n}\tilde\rho_\varepsilon(\tilde e_i, \tilde J\tilde e_i)$ at a point $x \in X$, where $(\tilde U, \Gamma, \varphi)$ is a local uniformising chart of X covering x, $\tilde x$ is a point of $\tilde U$ that is mapped to x under φ, $\tilde\rho_\varepsilon = \varphi^*\rho_\varepsilon$ as above, $\tilde J$ is the complex structure on $\tilde U \subseteq \mathbb{C}^n$, and $\{\tilde e_i\}_{i=1}^{2n}$ is an orthonormal basis of $T_{\tilde x}\tilde U$ with respect to $\varphi^*\omega$. Then, since $\int_{U(\varepsilon)}(\mathrm{tr}_\omega\,\rho_\varepsilon)\,\omega^n \leq 2n\int_X \rho_\varepsilon \wedge \omega^{n-1}$, and since this latter integral is bounded as $\varepsilon \longrightarrow 0$, we know that the integral $\int_{U(\varepsilon)}(\mathrm{tr}_\omega\,\rho_\varepsilon)\,\omega^n$ is bounded as $\varepsilon \longrightarrow 0$. Furthermore, recall from above that $\int_{U(\varepsilon)}\omega^n \geq C_0$. Putting these two facts together, we deduce the existence of an open subset $V(\varepsilon)$ of $U(\varepsilon)$ and of a constant $C > 0$ independent of ε such that

$$\mathrm{tr}_\omega\,\rho_\varepsilon \leq C \quad \text{on } V(\varepsilon). \tag{B.3}$$

We now derive a contradiction. Let $\varepsilon > 0$ and let $U(\varepsilon)$ and $V(\varepsilon)$ be as above. By Aronszajn's theorem, we know that ξ is not identically equal to zero on $V(\varepsilon)$. Hence there exists a point $x \in V(\varepsilon)$ and an isotropy chart $(\tilde U, \Gamma, \varphi)$ covering x such that $\varphi^*\xi \neq 0$ at the point $\tilde x := \varphi^{-1}(x)$. As before, let $(\theta_1, \ldots, \theta_n)$ be a local unitary frame of $\Lambda^{1,0}\tilde U$ with respect to the φ-pullback $\tilde\omega$ of ω in which the φ-pullback $\tilde\rho_\varepsilon$ of ρ_ε takes the form $\tilde\rho_\varepsilon = i\sum_j \lambda_j\theta_j \wedge \bar\theta_j$ at $\tilde x$. Also, for an ordered set $A = \{i_1, \ldots, i_q\}$,

write $\theta^A = \theta^{i_1} \wedge \ldots \wedge \theta^{i_q}$, and for $N = \{1, \ldots, n\}$, set $\varphi^*\xi = \sum_A \xi_A \theta^N \wedge \bar{\theta}^A$ for ξ_A some section of the line bundle $F_{\tilde{U}}$ over $\tilde{U}$ defining F for each A. Then at $\tilde{x}$, we have

$$\varphi^*(l(\rho_\varepsilon)\Lambda\xi) = \sum_A \left(\sum_{\alpha \in A} \lambda_\alpha \right) \xi_A \theta^N \wedge \bar{\theta}^A, \tag{B.4}$$

and by (B.3) above, we have the bound $\lambda_j \leq C$ for each $j = 1, \ldots, n$. By shrinking $\tilde{U}$ if necessary, we may assume that there exists an index A_0 such that ξ_{A_0} never vanishes on $\tilde{U}$ and such that $|\xi_A|_h \leq |\xi_{A_0}|_h$ at the point $\tilde{x}$ for any A. Here, $|\cdot|_h$ denotes the pointwise norm with respect to the Hermitian metric on $F_{\tilde{U}}$ defining h . Then we have that $\varphi^*(\langle \xi, \xi \rangle) \leq \binom{n}{q} |\xi_{A_0}|_h^2$ at $\tilde{x}$, and in light of (B.4), we also have that $\left(\sum_{i \in A_0} \lambda_i\right) |\xi_{A_0}|_h^2 \leq \varphi^*(\langle l(\rho_\varepsilon)\Lambda\xi, \xi \rangle)$ at $\tilde{x}$. Recalling the fact that $x \in U(\varepsilon)$ so that $u_\varepsilon(x) \leq 2\varepsilon q$, we thus see that for all $\alpha \in A_0$, $\lambda_\alpha \leq C\varepsilon$ at $\tilde{x}$, where we choose C (as in (B.3)) larger independently of ε if necessary. As a consequence, we find that $\prod_{j=1}^n \lambda_j \leq C^n \varepsilon^q$. By (B.2), it now follows that $0 < a_0 \leq C^n \varepsilon^q$. Since $q > 0$, this is clearly a contradiction for ε sufficiently small.

Appendix C. Comments on a paper by van Coevering

The main result in [41] constructs AC Calabi-Yau metrics on quasi-projective manifolds $X\backslash D$ under certain technical conditions that are less restrictive than the ones in Tian-Yau [40], but still more restrictive than those of Theorem 1.15. The key to the improvements over [40] made in [41] was to bring in tools from the theory of 1-convex complex manifolds; cf. [18, Appendix A].

However, as we will show in this section, an extra assumption was made during the proof in [41] *which forces $X\backslash D$ to be a crepant resolution of a cone*, thus formally reducing [41] to [43]. This extra assumption was needed to ensure that the Ricci potential of the approximating metric decays sufficiently fast. We circumvented this issue in our work by improving the analysis; cf. [18].

Let X be a compact Kähler manifold and let D be a smooth divisor in X such that the normal bundle N_D is positive on D. Denote by $\mathcal{I}_D$ and $\tilde{\mathcal{I}}_D$ the ideal sheaves of $D \subset X$ and of $D \subset N_D$ respectively, and define the corresponding *k-th infinitesimal neighbourhoods*, $D(k) := (D, \mathcal{O}_X/\mathcal{I}_D^{k+1})$ and $\tilde{D}(k) := (D, \mathcal{O}_{N_D}/\tilde{\mathcal{I}}_D^{k+1})$, as in [1, Definition 1.2]. Notice that $\tilde{D}(0) = D(0) = D$.

On p. 17 of [41], van Coevering assumes that $D(1)$ is isomorphic to $\tilde{D}(1)$. This, together with the cohomology vanishing condition in the statement of his main theorem, is what eventually allows him to construct a k-th approximating metric whose Ricci potential is $O(r^{-\nu k})$ for some $\nu > 0$. The following proposition will show, however, that all these conditions together already force $X\backslash D$ to be a resolution of the normal Stein variety $(N_D^*)^\times$, the blowdown of the zero section (equivalently, the Remmert reduction; cf. [18, Appendix A]) of the total space of N_D^*.

Proposition C.1. *Let X be a compact Kähler manifold of dimension $n \geq 3$. Let D be a smooth divisor in X such that N_D is positive, $D(1) \cong \tilde{D}(1)$, and $H^1(D, T_X|_D \otimes N_D^{-k}) = 0$ for every $k \geq 2$. Then $X\backslash D$ is 1-convex, and the Remmert reduction of $X\backslash D$ is biholomorphic to $(N_D^*)^\times$.*

The assumption that $D(1) \cong \tilde{D}(1)$, or equivalently, that the tangent sequence of D in X splits holomorphically [1, Proposition 1.5], is logically independent of the vanishing assumption. Notice in particular that vanishing up to some $k_0 \gg 1$ would be enough for the purposes of [41]; however, the vanishing may very well hold for all $k \in \mathbb{N}$ and yet the $D(1) \cong \tilde{D}(1)$ condition may fail.

Example C.2. (i) For $X = \mathrm{Bl}_p\mathbb{C}P^3$ and D the strict transform of a smooth quadric through p, the vanishing holds according to [41, Example 6.2] but the tangent sequence does not split. This follows from Proposition C.1 because $X\backslash D$ is affine. For an alternative proof, one could also appeal to the classification of submanifolds of X with a split tangent sequence in [25, Theorem 6.5].

(ii) For X a quadric and D a hyperplane section, the tangent sequence splits [25, Theorem 4.7] but the vanishing cannot hold for all k, again by Proposition C.1 and because $X\backslash D$ is affine.

Proof of Proposition C.1. Tensoring the tangent sequence $0 \longrightarrow T_D \longrightarrow T_X|_D \longrightarrow N_D \longrightarrow 0$ with the sheaf of sections of N_D^{-k} yields the following exact sequence of cohomology:

$$H^0(D, N_D^{1-k}) \longrightarrow H^1(D, T_D \otimes N_D^{-k}) \longrightarrow H^1(D, T_X|_D \otimes N_D^{-k}) \longrightarrow H^1(D, N_D^{1-k}).$$

Since N_D is ample and $n \geq 3$, the Kodaira vanishing theorem shows that

$$H^1(D, T_D \otimes N_D^{-k}) = H^1(D, T_X|_D \otimes N_D^{-k}) = 0$$

for $k \geq 2$. We may now lift the assumed isomorphism $D(1) \cong \tilde{D}(1)$ to an isomorphism $D(k) \cong \tilde{D}(k)$ for every k [1, Remark 4.2]. So far, the argument follows [41].

However, in our setting, having a formal isomorphism to every order is enough to conclude that X must be isomorphic to the total space of N_D in a neighbourhood of D. Indeed, by [16, Satz 4], N_D merely needs to admit a Hermitian metric whose curvature has at least one positive eigenvalue.

Since N_D is positive, $X\backslash D$ is 1-convex. Thus, $X\backslash D$ admits a Remmert reduction V, which is then isomorphic to $(N_D^*)^\times$ away from compact sets. Both V and $(N_D^*)^\times$ are normal Stein varieties, and normality implies Hartogs extension [36, Theorem 6.6], so the rings of entire functions on V and on $(N_D^*)^\times$, and hence the varieties themselves [23, Chapter V, §7], are isomorphic. $\square$

References

1. M. Abate, F. Bracci, and F. Tovena, *Embeddings of submanifolds and normal bundles*, Adv. Math. **220** (2009), no. 2, 620–656. MR 2466428 (2010f:32009)
2. W. L. Baily, *The decomposition theorem for V-manifolds*, Amer. J. Math. **78** (1956), 862–888. MR 0100103 (20 #6537)
3. ———, *On the imbedding of V-manifolds in projective space*, Amer. J. Math. **79** (1957), 403–430. MR 0100104 (20 #6538)
4. E. Barletta, S. Dragomir, and K. L. Duggal, *Foliations in Cauchy-Riemann geometry*, Mathematical Surveys and Monographs, vol. 140, American Mathematical Society, Providence, RI, 2007. MR 2319199 (2008j:32047)
5. J. Bertin, J.-P. Demailly, L. Illusie, and C. Peters, *Introduction to Hodge theory*, SMF/AMS Texts and Monographs, vol. 8, American Mathematical Society, Providence, RI, 2002, Translated from the 1996 French original by James Lewis and Peters. MR 1924513 (2003g:14009)
6. R. Blache, *Chern classes and Hirzebruch-Riemann-Roch theorem for coherent sheaves on complex-projective orbifolds with isolated singularities*, Math. Z. **222** (1996), no. 1, 7–57. MR 1388002 (97d:14015)
7. J. E. Borzellino, *Orbifolds with lower Ricci curvature bounds*, Proc. Amer. Math. Soc. **125** (1997), no. 10, 3011–3018. MR 1415575 (98e:53060)
8. R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York, 1982. MR 658304 (83i:57016)
9. C. P. Boyer and K. Galicki, *The twistor space of a 3-Sasakian manifold*, Internat. J. Math. **8** (1997), no. 1, 31–60. MR 1433200 (98e:53072)
10. ———, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008. MR 2382957 (2009c:53058)
11. E. Calabi, *Métriques kählériennes et fibrés holomorphes*, Ann. Sci. École Norm. Sup. (4) **12** (1979), no. 2, 269–294. MR 543218 (83m:32033)
12. F. Campana and B. Claudon, *Abelianity conjecture for special compact kähler threefolds*, arXiv:1107.0168v1 [math.AG] (2007).
13. S. Cappell, D. DeTurck, H. Gluck, and E. Y. Miller, *Cohomology of harmonic forms on Riemannian manifolds with boundary*, Forum Math. **18** (2006), no. 6, 923–931. MR 2278607 (2008c:58003)
14. X. X. Chen and G. Tian, *Ricci flow on Kähler-Einstein manifolds*, Duke Math. J. **131** (2006), no. 1, 17–73. MR 2219236 (2007c:53083)
15. Y.-J. Chiang, *Harmonic maps of V-manifolds*, Ann. Global Anal. Geom. **8** (1990), no. 3, 315–344. MR 1089240 (92c:58021)
16. M. Commichau and H. Grauert, *Das formale Prinzip für kompakte komplexe Untermannigfaltigkeiten mit 1-positivem Normalenbündel*, Ann. of Math. Stud., vol. 100, Princeton Univ. Press, Princeton, N.J., 1981, pp. 101–126. MR 627752 (84a:32018)
17. R. J. Conlon, *On the construction of asymptotically conical Calabi-Yau manifolds*, Ph.D. thesis, Imperial College London, 2011, published online at http://hdl.handle.net/10044/1/8991.
18. R. J. Conlon and H.-J. Hein, *Asymptotically conical Calabi-Yau manifolds, I*, arXiv:1205.6347v1 [math.DG] (2012).
19. J.-P. Demailly and T. Peternell, *A Kawamata-Viehweg vanishing theorem on compact Kähler manifolds*, J. Differential Geom. **63** (2003), no. 2, 231–277. MR 2015548 (2004m:32040)
20. I. Enoki, *Kawamata-Viehweg vanishing theorem for compact Kähler manifolds*, Einstein metrics and Yang-Mills connections (Sanda, 1990), Lecture Notes in Pure and Appl. Math., vol. 145, Dekker, New York, 1993, pp. 59–68. MR 1215279 (94f:32063)
21. C. L. Epstein and G. M. Henkin, *Stability of embeddings for pseudoconcave surfaces and their boundaries*, Acta Math. **185** (2000), no. 2, 161–237. MR 1819994 (2002e:32048)
22. H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368. MR 0137127 (25 #583)
23. H. Grauert and R. Remmert, *Theory of Stein spaces*, Classics in Mathematics, Springer-Verlag, Berlin, 2004, Translated from the German by Alan Huckleberry, Reprint of the 1979 translation. MR 2029201 (2004k:32015)
24. P. A. Griffiths, *The extension problem in complex analysis. II. Embeddings with positive normal bundle*, Amer. J. Math. **88** (1966), 366–446. MR 0206980 (34 #6796)
25. P. Jahnke, *Submanifolds with splitting tangent sequence*, Math. Z. **251** (2005), no. 3, 491–507. MR 2190340 (2007a:14049)
26. V. S. Kulikov, *Mixed Hodge structures and singularities*, Cambridge Tracts in Mathematics, vol. 132, Cambridge University Press, Cambridge, 1998. MR 1621831 (99d:14009)
27. C. LeBrun, *Fano manifolds, contact structures, and quaternionic geometry*, Internat. J. Math. **6** (1995), no. 3, 419–437. MR 1327157 (96c:53108)

28. E. Lerman and S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 4201–4230. MR 1401525 (98a:57043)
29. B. H. Lian and S.-T. Yau, *Period Integrals of CY and General Type Complete Intersections*, arXiv:1105.4872v4 [math.AG] (2011).
30. X. Ma, *Orbifolds and analytic torsions*, Trans. Amer. Math. Soc. **357** (2005), no. 6, 2205–2233 (electronic). MR 2140438 (2006i:58046)
31. D. Martelli, J. Sparks, and S.-T. Yau, *Sasaki-Einstein manifolds and volume minimisation*, Comm. Math. Phys. **280** (2008), no. 3, 611–673. MR 2399609 (2009d:53054)
32. J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963. MR 0163331 (29 #634)
33. I. Moerdijk and D. A. Pronk, *Simplicial cohomology of orbifolds*, Indag. Math. (N.S.) **10** (1999), no. 2, 269–293. MR 1816220 (2002b:55012)
34. J. R. Munkres, *Elementary differential topology*, Lectures given at Massachusetts Institute of Technology, Fall, vol. 1961, Princeton University Press, Princeton, N.J., 1966. MR 0198479 (33 #6637)
35. O. Riemenschneider, *A generalization of Kodaira's embedding theorem*, Math. Ann. **200** (1973), 99–102. MR 0326009 (48 #4355)
36. H. Rossi, *Vector fields on analytic spaces*, Ann. of Math. (2) **78** (1963), 455–467. MR 0162973 (29 #277)
37. I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363. MR 0079769 (18,144a)
38. Y. T. Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. **27** (1974), 53–156. MR 0352516 (50 #5003)
39. J. Sparks, *Sasaki-Einstein manifolds*, Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics, Surv. Differ. Geom., vol. 16, Int. Press, Somerville, MA, 2011, pp. 265–324. MR 2893680 (2012k:53082)
40. G. Tian and S.-T. Yau, *Complete Kähler manifolds with zero Ricci curvature. II*, Invent. Math. **106** (1991), no. 1, 27–60. MR 1123371 (92j:32028)
41. C. van Coevering, *A construction of complete Ricci-flat Kähler manifolds*, arXiv:0803.0112v5 [math.DG] (2008).
42. ———, *Regularity of asymptotically conical Ricci-flat Kähler metrics*, arXiv:0912.3946v5 [math.DG] (2010).
43. ———, *Ricci-flat Kähler metrics on crepant resolutions of Kähler cones*, Math. Ann. **347** (2010), no. 3, 581–611. MR 2640044
44. ———, *Examples of asymptotically conical Ricci-flat Kähler manifolds*, Math. Z. **267** (2011), no. 1-2, 465–496. MR 2772262 (2012d:53134)

Department of Mathematics & Statistics, McMaster University, Hamilton, ON L8S 4K1, Canada
E-mail address: `rconlon@math.mcmaster.ca`

Department of Mathematics, Imperial College, London SW7 2AZ, United Kingdom
E-mail address: `h.hein@imperial.ac.uk`